

Math 39100: Feb. 27, 2023: LECTURE 9

Exam 1: Monday, March 6 (1.2, 1.3, CH2, 3.1, 3.2, 3.3, 3.4)

last time: S.O.L.D.E with constant coeff: (Homog)

$$ay'' + by' + cy = 0$$

$$\text{Soln: } y = e^{rt}$$

$$\text{Char poly: } ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$b^2 - 4ac > 0$$

$r_1 \neq r_2$ (Real distinct roots)

$$y = C_1 \underbrace{e^{r_1 t}}_{y_1} + C_2 \underbrace{e^{r_2 t}}_{y_2}$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

(i) y_1 and y_2 are linearly indep.

(ii) $\{y_1, y_2\}$ is a fundamental set of solutions.

(iii) $y = C_1 y_1 + C_2 y_2$ is the general solution.

Abel's Theorem:
$$W = C e^{-\int p(t) dt}$$

§ 3.3 Complex Roots of the Characteristic Polynomial

Case 2: $b^2 - 4ac < 0$

$b^2 < 4ac \Rightarrow$ Roots are Complex conjugate

$$r_1 = \alpha + \beta i$$

$$r_2 = \alpha - \beta i$$

$$r = \alpha \pm \beta i$$

Soln: $y = C e^{rt} = C e^{(\alpha \pm \beta i)t}$

$$= C e^{\alpha t} e^{\pm \beta i t}$$

Euler's Formulas:

Recall:
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

Note:

$$i = \sqrt{-1} \quad i^2 = -1$$

$$i^2 = -1 \quad !$$

$$i^3 = -i$$

$$i^4 = 1$$

$$= 1 + \frac{ix}{1!} + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \frac{i^4 x^4}{4!} + \frac{i^5 x^5}{5!} + \dots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \dots$$

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)}_{\cos(x)} + i \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)}_{\sin(x)}$$

$$e^{ix} = \cos(x) + i \sin(x)$$

← Euler's Formula

$$r = \alpha \pm \beta i$$

$$\begin{aligned} \text{Soln: } y &= A e^{\alpha t} e^{i\beta t} = A e^{\alpha t} (\cos(\beta t) + i \sin(\beta t)) \\ &= A e^{\alpha t} \underbrace{\cos(\beta t)}_{y_1} + B i e^{\alpha t} \underbrace{\sin(\beta t)}_{y_2} \end{aligned}$$

⇒

$$\begin{aligned} \text{Gen. Soln: } y &= C_1 y_1 + C_2 y_2 \\ &= C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t) \end{aligned}$$

$$y = e^{\alpha t} (C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

$$r_1 = \alpha + \beta i \quad \text{and} \quad r_2 = \alpha - \beta i$$

$$y_1 = e^{r_1 t} = e^{(\alpha + \beta i)t}$$

$$y_2 = e^{r_2 t} = e^{(\alpha - \beta i)t}$$

$$y = Ay_1 + By_2$$

$$= A e^{(\alpha + \beta i)t} + B e^{(\alpha - \beta i)t}$$

$$= A e^{\alpha t} \cdot e^{i\beta t} + B e^{\alpha t} \cdot e^{-i\beta t}$$

$$= A e^{\alpha t} (\cos(\beta t) + i \sin(\beta t))$$

$$+ B e^{\alpha t} (\cos(\beta t) - i \sin(\beta t))$$

$$= A e^{\alpha t} \cos(\beta t) + B e^{\alpha t} \cos(\beta t)$$

$$+ i A e^{\alpha t} \sin(\beta t) - B i e^{\alpha t} \sin(\beta t)$$

$$= \underbrace{(A + B)}_{C_1} e^{\alpha t} \cos(\beta t) + \underbrace{(A i - B i)}_{C_2} e^{\alpha t} \sin(\beta t)$$

$$y = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)$$

Solve $y'' + y = 0$.

Char. poly: $r^2 + 1 = 0$

$$r = \pm i = 0 \pm i$$

$$\alpha = 0$$

$$\beta = 1$$

$$y = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)$$

$$y = C_1 \cos(t) + C_2 \sin(t)$$

eg. solve $y'' - 2y' + 2y = 0$.

Char. poly: $r^2 - 2r + 2 = 0$

$$r = \frac{+2 \pm \sqrt{4 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm i2}{2}$$

$$= 1 \pm i$$

\uparrow \uparrow
 α β

$$y = e^t (C_1 \cos(t) + C_2 \sin(t))$$

IVP: $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = -1$.

char. poly: $r^2 + 4 = 0$

$$r = \pm 2i = 0 \pm 2i$$

\uparrow \uparrow
 α β

$$y = C_1 \cos(2t) + C_2 \sin(2t)$$

$$y' = -2C_1 \sin(2t) + 2C_2 \cos(2t)$$

$$y(0) = 1 \Rightarrow 1 = C_1$$

$$y'(0) = -1 \Rightarrow -1 = 2C_2 \Rightarrow C_2 = -\frac{1}{2}$$

Soln: $y = \cos(2t) - \frac{1}{2} \sin(2t)$

§ 3.4 Repeated Roots ; Reduction of Order

Case 3: $b^2 - 4ac = 0$

$$r = -\frac{b}{2a} \quad \text{one root.}$$

What is the general solution?

Solve the differential equation

$$y'' + 4y' + 4y = 0.$$

Char. poly: $r^2 + 4r + 4 = 0$

$$(r+2)^2 = 0$$

$$r_1 = -2 = r_2 \quad \text{repeated root.}$$

$$y_1 = e^{r_1 t} = e^{-2t}$$

Q. what is the second indep. soln?

Using D'Alembert's method we seek the second indep.

Soln: Assume that the
general soln $y = c y_1$

let $c = v(t)$

a function of
 t .

$$y = v(t) y_1$$

$$y = v(t) e^{-2t} = v e^{-2t}$$

$$y' = v' e^{-2t} - 2v e^{-2t}$$

$$y'' = v'' e^{-2t} - 2v' e^{-2t} - 2v' e^{-2t} + 4v e^{-2t}$$

$$y'' = v'' e^{-2t} - 4v' e^{-2t} + 4v e^{-2t}$$

$$y'' + 4y' + 4y = 0.$$

plus in y , y' and y'' into given eq.

$$v'' e^{-2t} - 4v' e^{-2t} + 4v e^{-2t} + 4v' e^{-2t} - 8v e^{-2t} + 4v e^{-2t} = 0$$

$$v'' e^{-2t} = 0$$

$$\Rightarrow v'' = 0$$

$$\Rightarrow v' = c_1$$

$$\Rightarrow v = c_1 t + c_2$$

$$\text{Soln: } y = v y_1 \quad \text{where } y_1 = e^{-2t}$$

$$y = (C_1 t + C_2) e^{-2t}$$

$$\Rightarrow y = C_1 t e^{-2t} + C_2 e^{-2t}$$

For Repeated roots: the general solution:

$$y = C_1 t y_1 + C_2 y_1$$

Find the solution of the initial value problem

$$y'' - y' + \frac{y}{4} = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{3}.$$

$$\text{Char. poly: } r^2 - r + \frac{1}{4} = 0$$

$$\left(r - \frac{1}{2}\right)^2 = 0$$

$$r_1 = r_2 = \frac{1}{2}$$

repeated root.

$$y = C_1 e^{\frac{1}{2}t} + C_2 t e^{\frac{1}{2}t}$$

$$y' = \frac{1}{2} C_1 e^{\frac{1}{2}t} + C_2 e^{\frac{1}{2}t} + \frac{1}{2} C_2 t e^{\frac{1}{2}t}$$

$$y(0) = 2 \Rightarrow C_1 = 2$$

$$y'(0) = \frac{1}{3} \Rightarrow \frac{1}{3} = \frac{1}{2}(2) + C_2$$

$$\frac{1}{3} - 1 = C_2 \Rightarrow C_2 = -\frac{2}{3}$$

$$y = 2e^{\frac{1}{2}t} - \frac{2}{3}te^{\frac{1}{2}t}$$

Summary: S.O. C. P.E with Constant Coeff. homogeneous:

3.1 (i) if $r_1 \neq r_2 \in \mathbb{R}$: Gen. Soln : $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

3.3 (ii) if $r_1, r_2 \in \mathbb{C}$: Gen. Soln : $y = e^{\alpha t} (C_1 \cos(\beta t) + C_2 \sin(\beta t))$

3.4 (iii) if $r_1 = r_2 \in \mathbb{R}$: Gen. Soln : $y = C_1 e^{rt} + C_2 t e^{rt}$

Next time: Reduction of Order.