Math 39100 : Mar. 20. 2023 : LECTURE 14

§ 3.7 / 3.8 CONT.

Recall: $\boxed{m}u'' + \boxed{\gamma}u' + \boxed{k}u = G(t)$

$m = \frac{W}{g}$ $g = 9.8\ m/s^2$ or $32ft/s^2$

$\gamma = \frac{\text{Resist. force}}{\text{Velocity}}$

$k = \frac{W}{L}$

$u(0)$ = initial position

$u'(0)$ = initial velocity

A weight of $\frac{1}{10}N$ stretches a spring 5cm (1/20 m). If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/s($\frac{1}{10}$ m/s), and if there is no damping, at what time does the mass first return to its equilibrium position?

$m = ?$ $W = mg \Rightarrow \frac{W}{g} = m$

$\boxed{\gamma = 0}$ $\frac{\frac{1}{10}}{9.8} = m \Rightarrow \boxed{m = \frac{1}{98}}$

$k = ?$

$G(t) = 0$ $W = kL \Rightarrow \boxed{k} = \frac{W}{L} = \frac{\frac{1}{10}}{\frac{1}{20}} = \boxed{2}$

$L = \frac{1}{20}$

$mu'' + ku = 0$

$$\boxed{\frac{1}{98}u'' + 2u = 0, \quad u(0) = 0, \quad u'(0) = \frac{1}{10} \text{ m/s}}$$

$$u'' + 196u = 0$$

$$r^2 + 196 = 0$$

$$r = \pm\sqrt{196}\, i = \pm 14i$$

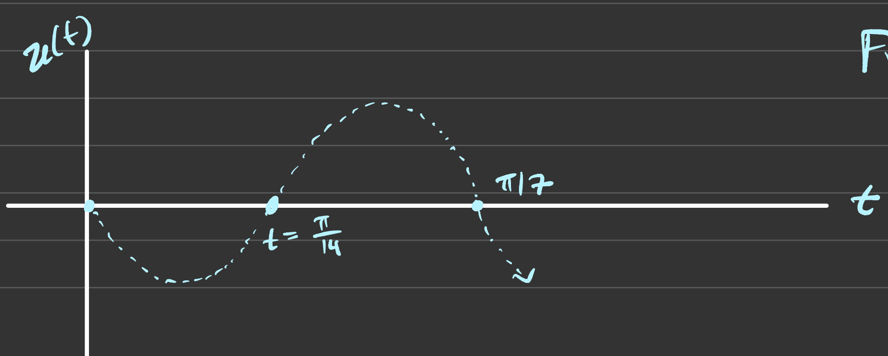
$$u(t) = C_1 \cos(14t) + C_2 \sin(14t) = C_2 \sin(14t)$$

$$u(0) = 0 \Rightarrow \boxed{C_1 = 0}$$

$$u'(t) = 14C_2 \cos(14t)$$

$$u'(0) = \frac{1}{10} \Rightarrow \frac{1}{10} = 14C_2 \Rightarrow \boxed{C_2 = \frac{1}{140}}$$

$$\Rightarrow \boxed{u(t) = \frac{1}{140}\sin(14t)}$$



Frequency $= \omega$
$= 14$

Period $= \frac{2\pi}{\omega}$
$= \frac{2\pi}{14}$

When $t = \frac{\pi}{14}$, the mass first returns to $= \frac{\pi}{7}$

its equilibrium position.

Free Oscillation: No damping or No External force:

(simple harmonic motion)

$$mu'' + ku = 0$$

$$u'' + \frac{k}{m} u = 0$$

$$r = \pm \sqrt{\frac{k}{m}}\, i \qquad \text{let } \sqrt{\frac{k}{m}} = \omega.$$

$$u(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

Using Sum of two angles identity we obtain,

$$u(t) = A \cos(\omega t) + B \sin(\omega t)$$

$$\boxed{\text{Amplitude} = R = \sqrt{A^2 + B^2}}$$

$\boxed{\text{Frequency} : \omega}$ and $\boxed{\text{period} = \frac{2\pi}{\omega}}$

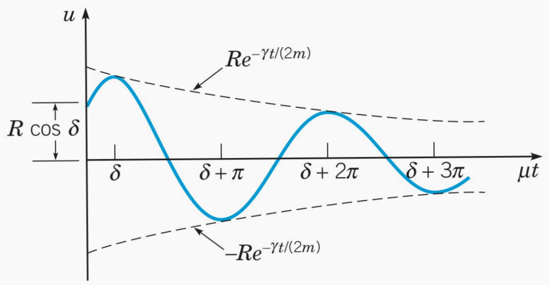


FIGURE 3.7.5 Damped vibration; $u = Re^{-\gamma t/2m}\cos(\mu t - \delta)$. Note that the scale for the horizontal axis is μt.

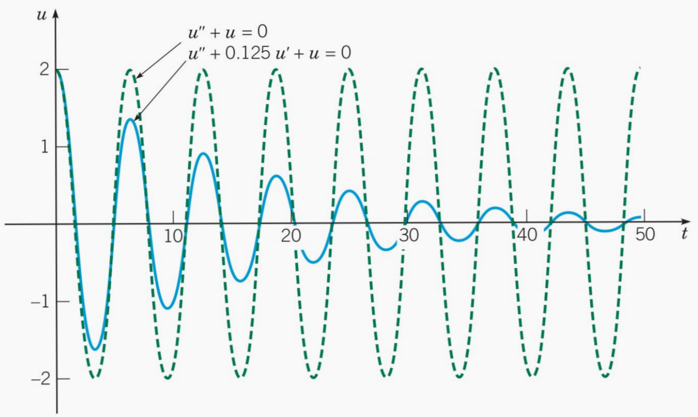


FIGURE 3.7.7 Vibration with small damping (solid blue curve) and with no damping (dashed green curve). Both motions have the same initial conditions: $u(0) = 2$, $u'(0) = 0$.

8. A hanging spring is stretched 18 inches (1.5 feet) by a mass with weight 60 pounds. Set up the initial value problem (differential equation and initial conditions) which describes the motion, neglecting frictions, when the weight starts from rest at the equilibrium position and is subjected to an external force of $5\cos(7t)$. (Recall that g, the acceleration due to gravity is $32ft/sec^2$.)

$G(t)$

$$mu'' + \cancel{\gamma u'} + ku = G(t),$$

$$u(0) = 0$$

$$u'(0) = 0$$

$$\boxed{m = \frac{w}{g} = \frac{60}{32} = \frac{15}{8}}$$

$\gamma = 0$

$$k = \frac{w}{L} = \frac{60}{1.5} = 40$$

$$\boxed{\frac{15}{8} u'' + 40u = 5\cos(7t), \quad u(0) = u'(0) = 0}$$

CH4: Higher Order D.E (4.2 and 4.3)

4.2 Higher Order linear Homoge. D.E. with Constant Coeff:

Consider the n^{th} order linear homogeneous differential equation

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0, \qquad (1)$$

where $a_0, a_1, \ldots, a_n$ are real constants and $a_0 \neq 0$. From our knowledge of second-order linear equations with constant coefficients, it is natural to anticipate that $y = e^{rt}$ is a solution

Find the general solution of

$$y^{(4)} - y = 0.$$

Char. Poly: $r^4 - 1 = 0$

$$(r^2+1)(r^2-1) = 0$$

$$\underbrace{(r^2+1)}_{r_1 = r_2 = \pm i}\underbrace{(r+1)}_{r_3 = -1}\underbrace{(r-1)}_{r_4 = 1} = 0$$

$$\boxed{y = \underbrace{C_1 \cos(t)}_{y_1} + \underbrace{C_2 \sin(t)}_{y_2} + \underbrace{C_3 e^{-t}}_{y_3} + \underbrace{C_4 e^{t}}_{y_4}}$$

Also find the solution that satisfies the initial conditions

$$y(0) = \frac{7}{2}, \quad y'(0) = -4, \quad y''(0) = \frac{5}{2}, \quad y'''(0) = -2$$

Finish me...

Find the general solution of

$$y^{(4)} + 2y'' + y = 0.$$

Char. poly: $r^4 + 2r^2 + 1 = 0$

<u>Rational Root Test:</u> IF $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

Then the <u>possible</u> roots are :

$$\frac{p}{q} = \pm \frac{\text{factors of } a_0}{\text{factors of } a_n}$$

$a_0 = 1$, $a_n = 1$ possible roots : $\{\pm 1\}$ → these are not roots.

$$r^4 + 2r^2 + 1 = 0$$

$$(r^2+1)(r^2+1) = 0$$

$$(r^2+1)^2 = 0$$

$$r = \pm i \quad (\text{repeated})$$

$$\boxed{y = C_1 \cos(t) + C_2 \sin(t) + C_3 t \cos(t) + C_4 t \sin(t)}$$