

Math 39100 : Mar. 22. 2023 : LECTURE 15

§ 4.2 cont.

Find the general solution of $y''' + y = 0$.

Char. poly: $r^3 + 1 = 0$

r_1, r_2, r_3 ?

$$(r+1)(r^2 - r + 1) = 0$$

$$r^3 = -1$$

$$\begin{aligned} \textcircled{\text{A}} \quad a^3 + b^3 &= (a+b)(a^2 - ab + b^2) \\ \textcircled{\text{B}} \quad a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \end{aligned}$$

$$r_1 = \sqrt[3]{-1} = -1$$

De Moivre's Theorem

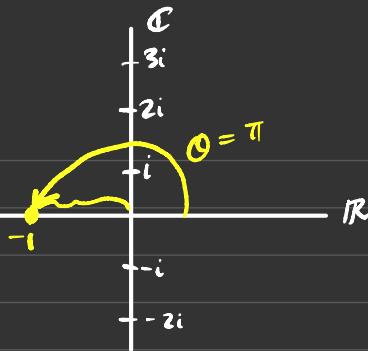
If $z = \alpha + \beta i = Re^{i\theta}$ then $z^n = R^n e^{in\theta}$.

$$\text{Recall: } e^{ix} = \cos(x) + i\sin(x)$$

$$= R^n (\cos(n\theta) + i\sin(n\theta))$$

We have $r^3 + 1 = 0$

$$\begin{aligned} r^3 = -1 &= Re^{i\theta} & R = ? \\ &= 1 \cdot e^{i(\pi + 2n\pi)} & \theta = ? \end{aligned}$$



$n = 0, 1, 2.$

$$r^3 = e^{i(\pi + 2n\pi)}$$

$$r = e^{i\left(\frac{\pi}{3} + \frac{2n\pi}{3}\right)}, n = 0, 1, 2.$$

$$n=0, r = e^{i\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$$

$$r_2 = r_1 = \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$n=1, r = e^{i\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)} = e^{i(\pi)}$$

$$= \cos(\pi) + i\sin(\pi)$$

$$r_3 = [-1]$$

Gen. Soln : $y = C_1 e^{i\sqrt{3}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 e^{i\sqrt{3}t} \sin\left(\frac{\sqrt{3}}{2}t\right) + C_3 e^{-t}$

In general, $r^n = Re^{i(\theta + 2k\pi)}$, then

$$r = R \cdot e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}$$

where $k = 0, 1, 2, \dots, n-1$.

4.3 Higher Order DE : Nonhomogeneous (Unde. Coeff.)

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t)$$

Poly or Expo or Sine or
Cosine.

Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^t.$$

$$y_h: \quad y''' - 3y'' + 3y' - y = 0$$

Char. Eq: $r^3 - 3r^2 + 3r - 1 = 0$

$r_1 = ?$
 $r_2 = ?$

$r_3 = ?$

1 1 1 pascal's triangle

$$(r-1)^3 = 0$$

Algebra: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

1 3 3 1

$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

1 4 6 4 1

$r_1 = r_2 = r_3 = 1$

$$y_h = C_1 e^t + C_2 t e^t + C_3 t^2 e^t$$

$$y_p = Ae^t \cdot t^3 = At^3e^t$$

$$y_p' = (3At^2e^t + At^3e^t)$$

$$\begin{aligned} y_p'' &= 6At^2e^t + 3At^2e^t + 3At^2e^t + At^3e^t \\ &= (6At^2e^t + 6At^2e^t + At^3e^t) \end{aligned}$$

$$\begin{aligned} y_p''' &= 6Ae^t + 6At^2e^t + 12At^2e^t + 6At^2e^t + 3At^2e^t \\ &\quad + At^3e^t \\ &= (6Ae^t + 18At^2e^t + 9At^2e^t + At^3e^t) \end{aligned}$$

$$y''' - 3y'' + 3y' - y = 4e^t.$$

$$\begin{aligned} 6Ae^t + 18At^2e^t + 9At^2e^t + At^3e^t - 18At^2e^t - 18At^2e^t - 3At^3e^t \\ + 9At^2e^t + 3At^3e^t - At^3e^t = 4e^t \end{aligned}$$

$$6Ae^t = 4e^t \Rightarrow A = \frac{2}{3}$$

$$\text{Sln: } y = y_h + y_p$$

$$y = C_1 e^t + C_2 t e^t + C_3 t^2 e^t + \frac{2}{3} t^3 e^t$$

The form of the

Find a particular solution of

$$y''' - 4y' = t + 3 \cos t + e^{-2t}.$$

y_{P_1} y_{P_2} y_{P_3}

$$y_h: \quad y''' - 4y' = 0$$

$$\text{Char. poly: } r^3 - 4r = 0$$

$$r(r^2 - 4) = 0$$

$$r(r+2)(r-2) = 0$$

$$r_1 = 0, \quad r_2 = -2, \quad r_3 = 2$$

$$y_{P_1} = (A\epsilon + B)\cdot t$$

$$y_{P_2} = C \cos(t) + D \sin(t)$$

$$y_{P_3} = (E t e^{-2t}) \cdot t$$

$$y_h = C_1 + C_2 e^{2t} + C_3 e^{-2t}$$

$$y_p = y_{P_1} + y_{P_2} + y_{P_3}$$

$$y_p = A\epsilon^2 + B\epsilon t + C \cos(t) + D \sin(t) + E t e^{-2t}$$

12- For the equation $y^{(5)} - y'' = 7t^2 + e^t - \cos(\frac{\sqrt{3}}{2}t) + 5$, the general solution consists of y_h , general solution of the homogeneous given by (1), added to y_p , a particular solution obtained from the method of undetermined coefficients whose simplest form is given by (2):

$$y_h: \quad y^{(5)} - y'' = 0$$

$$\text{Char. Poly: } r^5 - r^2 = 0$$

$$r^2(r^3 - 1) = 0$$

$$r^2 = 0$$

$$r_1 = r_2 = 0$$

$$r^3 = 1 = e^{i(0+2n\pi)}, \quad n=0,1,2.$$

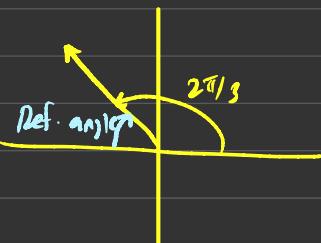


$$r = e^{i(\frac{2n\pi}{3})}$$

$$n=0 \quad r_3 = e^0 = 1$$

$$n=1, \quad r = e^{i(\frac{2\pi}{3})} = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$$

$$= -\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)$$



$$r_3 = r_4 = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$$

$$y_h = C_1 + C_2 t + C_3 e^t + C_4 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + C_5 e^{\frac{\sqrt{3}}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$r_1 = r_2 = 0$$

$$r=1$$

Complex conjugate

$$y^{(5)} - y'' = 7t^2 + e^t - \cos\left(\frac{\sqrt{3}}{2}t\right) + 5,$$

$$= \underbrace{7t^2 + 5}_{y_{P_1}} + \underbrace{e^t}_{y_{P_2}} - \underbrace{\cos\left(\frac{\sqrt{3}}{2}t\right)}_{y_{P_3}}$$

$$y_{p_1} = (A t^2 + B t + C) \cdot t^2$$

$$y_{p_2} = D e^t \cdot t$$

$$y_{p_3} = E \cos\left(\frac{\sqrt{3}}{2}t\right) + F \sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$y_p = y_{p_1} + y_{p_2} + y_{p_3}$$

$$y_p = A t^4 + B t^3 + C t^2 + D t e^t + E \cos\left(\frac{\sqrt{3}}{2}t\right) + F \sin\left(\frac{\sqrt{3}}{2}t\right)$$

Review Power Series