

Math 39100 : May. 8 . 2023 : LECTURE 24

Exam 3: (optional) Friday, May 12

Time: 4-5:15 pm

Room: Nac 1/511

Topics: CH2 (solving 1st order Eq.), CH3 (solving 2nd order eq.)
5.2, 10.2, 10.4 and 10.5.

Quiz 6: Monday, May 15 :

FINAL EXAM: May, 18 from 3:30 - 5:45 pm

Room: Nac 0/201

Cont. from last time:

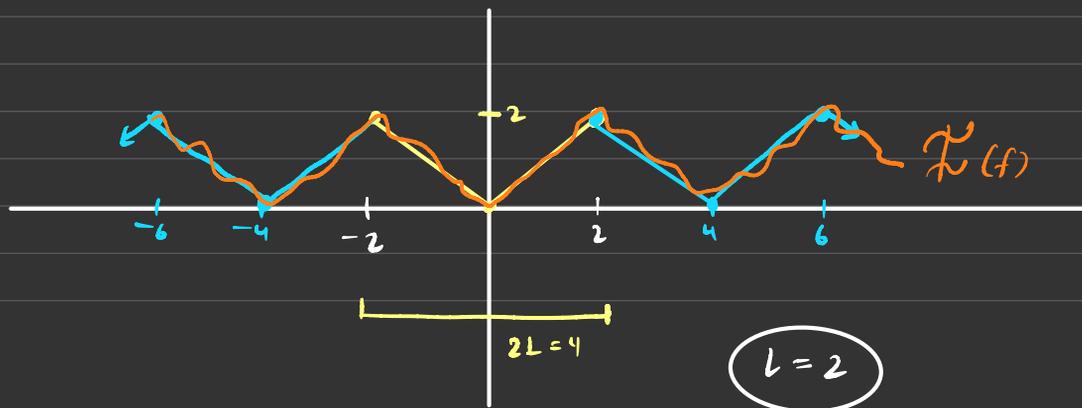
Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \leq x < 0, \\ x, & 0 \leq x < 2; \end{cases}$$

$$f(x+4) = f(x).$$

Determine the coefficients in this Fourier series.

$2L = 4 \Rightarrow L = 2$



$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx = \frac{1}{2} \int_{-2}^2 f(x) dx$$

$$= \frac{1}{2} \int_{-2}^0 -x dx + \frac{1}{2} \int_0^2 x dx$$

$$= \frac{2}{2} \int_0^2 f(x) dx \quad \leftarrow f \text{ is even}$$

$$= \int_0^2 x dx$$

$$= \boxed{2}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{2} \int_{-2}^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx$$

even · even = even func.

$$= \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx \quad \text{since } f \text{ is even}$$

$$= 1 \cdot \int_0^2 x \cdot \cos\left(\frac{n\pi x}{2}\right) dx$$

Tabular Method

$$= \frac{2}{n\pi} \times \sin\left(\frac{n\pi x}{2}\right) \Big|_0^2$$

$$+ \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right) \Big|_0^2$$

$$= \frac{4}{n^2\pi^2} \cos(n\pi) - \frac{4}{n^2\pi^2}$$

Der.		Inte
x	⊕	$\cos\left(\frac{n\pi x}{2}\right)$
1	⊖	$\frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$
0		$-\frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)$

$$a_n = \begin{cases} -\frac{8}{n^2\pi^2}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$\cos(n\pi) = (-1)^n$$

$$\frac{4(-1)^n}{n^2\pi^2} - \frac{4}{n^2\pi^2}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

since f is even

$$\tilde{f}(f) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

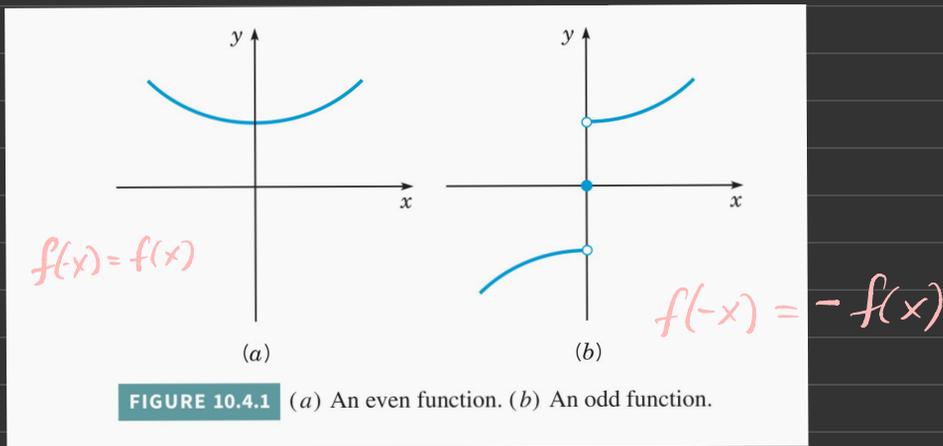
$$= \frac{2}{2} + \sum_{\substack{n=1 \\ (\text{odd})}}^{\infty} \frac{-8}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)$$

or

$$= 1 + \sum_{n=1}^{\infty} \frac{-8}{(2n-1)^2 \pi^2} \cos\left(\frac{(2n-1)\pi x}{2}\right)$$

10.4 Even and ODD FUNCTIONS

Given $f(x)$ on $[0, L]$ or $[-L, 0]$, then we can choose to extend $f(x)$ as an odd or even function.



$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right) + \sum b_n \sin\left(\frac{n\pi x}{L}\right)$$

For even extension : $b_n = 0$,

So, Fourier Cosine Series : $\tilde{f}_{\text{cosine}} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

$$a_0 = \frac{2}{L} \int_0^L f(x) dx \quad \text{and}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = 0$$

For odd Extension : $(a_n = 0 \text{ and } a_0 = 0)$

$f(x)$ has Fourier Sine series repre :

$$\tilde{f}_{\text{sine}} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

⊕ If $f(x)$ is neither odd nor even :

all Fourier coeff. a_0, a_n, b_n might survive.

Remarks: i) (even fun.) (even funct) = even fun.

ii) (eve fun) (odd funct) = odd fun.

iii) (odd fun) (odd funct) = even funct.

$$iv) \int_{-L}^L (\text{odd}) dx = 0$$

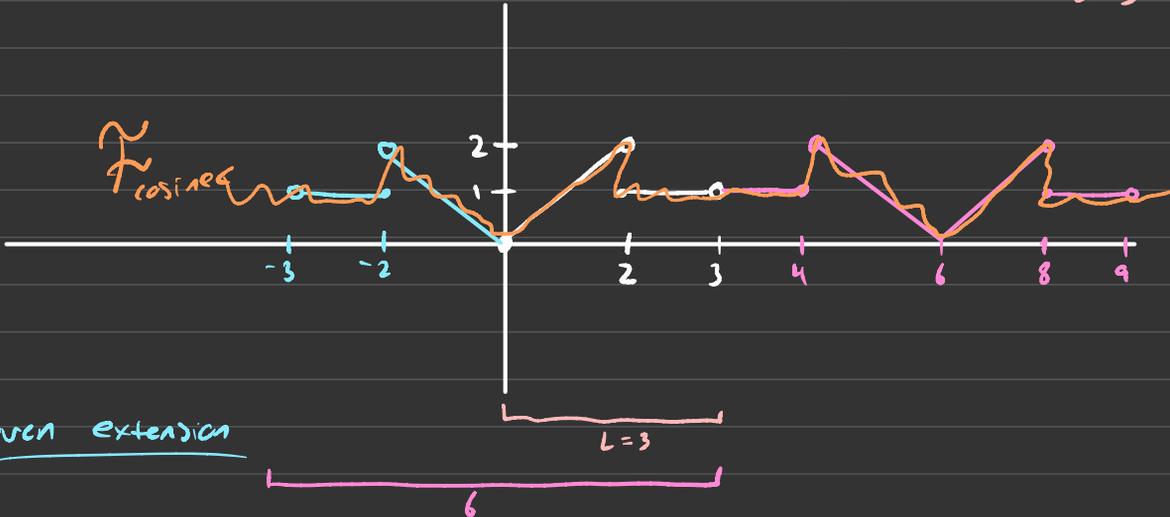
$$v) \int_{-L}^L (\text{even}) dx = 2 \int_0^L (\text{even}) dx$$

eg. Sketch the graphs of the even and odd extensions of f :

$$\text{Given } f(x) = \begin{cases} x, & 0 \leq x < 2 \\ 1, & 2 \leq x < 3 \end{cases}$$

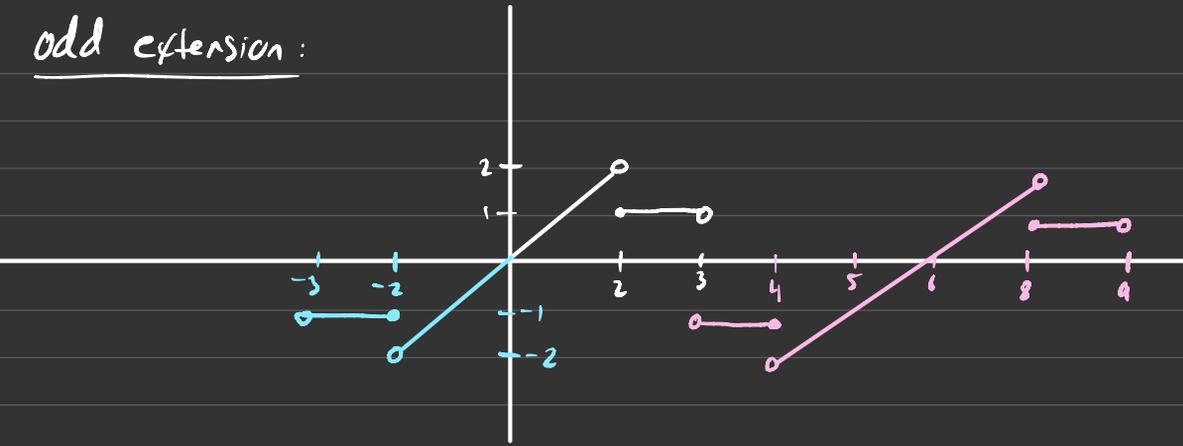
$$\text{When } f(x+6) = f(x).$$

$$2L = 6 \Rightarrow L = 3$$



$$f_{\text{cosine}} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right)$$

odd extension:



$$\mathcal{F}_{\text{Sine}} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right)$$

$$b_n = \frac{1}{3} \int_{-3}^3 f(x) \sin\left(\frac{n\pi x}{3}\right) dx$$

↙ odd

$$= \frac{2}{3} \int_0^3 f(x) \sin\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{2}{3} \left[\int_0^2 x \cdot \sin\left(\frac{n\pi x}{3}\right) dx + \int_2^3 1 \cdot \sin\left(\frac{n\pi x}{3}\right) dx \right]$$

$$= \frac{2}{3} \left[\frac{-3}{n\pi} x \cos\left(\frac{n\pi x}{3}\right) \right]_0^2$$

x	⊕	$\sin\left(\frac{n\pi x}{3}\right)$
1	⊖	$\frac{-3}{n\pi} \cos\left(\frac{n\pi x}{3}\right)$

$$+ \frac{9}{n^2 \pi^2} \sin\left(\frac{n\pi x}{3}\right) \Big|_0^2$$

$$0 \quad \downarrow - \frac{9}{n^2 \pi^2} \sin\left(\frac{n\pi x}{3}\right)$$

$$\left. - \frac{3}{n\pi} \cos\left(\frac{n\pi x}{3}\right) \Big|_2^3 \right]$$

$$= \frac{2}{3} \left[-\frac{6}{n\pi} \cos\left(\frac{2n\pi}{3}\right) + \frac{9}{n^2 \pi^2} \sin\left(\frac{2n\pi}{3}\right) - 0 \right.$$

$$\left. - \frac{3}{n\pi} \cos(n\pi) + \frac{3}{n\pi} \cos\left(\frac{2n\pi}{3}\right) \right]$$

$$\mathcal{F}_{\text{Sine}} = \sum_{n=1}^{\infty} \left[\frac{2}{3} \left(-\frac{3}{n\pi} \cos\left(\frac{2n\pi}{3}\right) + \frac{9}{n^2 \pi^2} \sin\left(\frac{2n\pi}{3}\right) - \frac{3}{n\pi} (-1)^n \right) \right] \sin\left(\frac{n\pi x}{3}\right)$$