

# Math 39100 : Feb. 21. 2023 : LECTURE 7

Exam 1: Monday, March 6

CH3: Second Order L.D.E with Constant Coefficients

§ 3.1: Homogeneous S.O.L.D.E.

$$\frac{d^2y}{dt^2} = f(t, y, y') \quad "t" \text{ indep.}$$

or  
"x"

Form:  $y'' + p(t)y' + q(t)y = g(t)$

if  $g(t) = 0$ , then we are looking at Homogeneous equations.

$g(t) \neq 0$ , it non-homogeneous.

Goal: Find solution to homogeneous S.O.L.D.E with constant coeff.

(i) Roots are real and distinct.

(ii) Roots are imaginary and real repeated (3.3 + 3.4)

eg. Solve  $y'' - 4y' + 3y = 0$ .

"ansatz"  
↳ "guess"

$$y = e^{rt}$$

$$r = ?$$

$$y' = re^{rt}$$

$$y'' = r^2 e^{rt}$$

plug  $y'$ ,  $y''$ ,  $y$  into given eq:

$$r^2 e^{rt} - 4re^{rt} + 3e^{rt} = 0$$

$$e^{rt} (r^2 - 4r + 3) = 0$$

$$r^2 - 4r + 3 = 0 \quad \leftarrow \text{Characteristic polynomial}$$

$$(r-3)(r-1) = 0$$

$$r-3=0 \quad | \quad r-1=0$$

$$r=3$$

$$r=1$$

← roots of the  
Char. Poly.

$$y_1 = e^{3t}, \quad y_2 = e^t$$

General soln:  $y = C_1 y_1 + C_2 y_2 \quad \leftarrow$  linear combination  
of  $y_1$  and  $y_2$ .

$$y = C_1 e^{3t} + C_2 e^{-t}$$

With two initial conditions, we can solve for  
 $C_1$  and  $C_2$ .

Solve the equation

$$y'' - y = 0.$$

Also find the solution that satisfies the initial conditions

$$y(0) = 2, \quad y'(0) = -1.$$

Solution form:  $y = e^{rt}$

Char. Poly:  $r^2 - 1 = 0$

$$r_1 = 1, \quad r_2 = -1$$

$$y_1 = e^t, \quad y_2 = e^{-t}$$

General solution:  $y = C_1 e^t + C_2 e^{-t} \rightarrow y' = C_1 e^t - C_2 e^{-t}$

$$y(0) = 2 \Rightarrow (2 = C_1 + C_2)$$

(5)

$$y'(0) = -1 \Rightarrow (-1 = C_1 - C_2)$$

Solve for  $C_1$  and  $C_2$ .

$$1 = 2C_1 \Rightarrow C_1 = \frac{1}{2}$$

$$C_2: 2 = \frac{1}{2} + C_2$$

$$2 - \frac{1}{2} = C_2$$

$$\frac{3}{2} = C_2$$

Specific Soln:

$$y = \frac{1}{2}e^t + \frac{3}{2}e^{-t}$$

Find the solution of the initial value problem

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 3.$$

$$\text{Char. poly : } r^2 + 5r + 6 = 0$$

$$(r+3)(r+2) = 0$$

$$r_1 = -3, \quad r_2 = -2$$

$$\text{Gen. Soln: } y = C_1 e^{rt} + C_2 e^{rt}$$

$$y = C_1 e^{-3t} + C_2 e^{-2t}; \quad y' = -3C_1 e^{-3t} - 2C_2 e^{-2t}$$

$$y(0) = 2 \Rightarrow \left\{ \begin{array}{l} 2 = C_1 + C_2 \end{array} \right.$$

$$y'(0) = 3 \Rightarrow \left\{ \begin{array}{l} 3 = -3C_1 - 2C_2 \end{array} \right.$$

$$C_1 = -7, \quad C_2 = 9$$

$$4 = 2C_1 + 2C_2$$

$$\oplus \quad 3 = -3C_1 - 2C_2$$

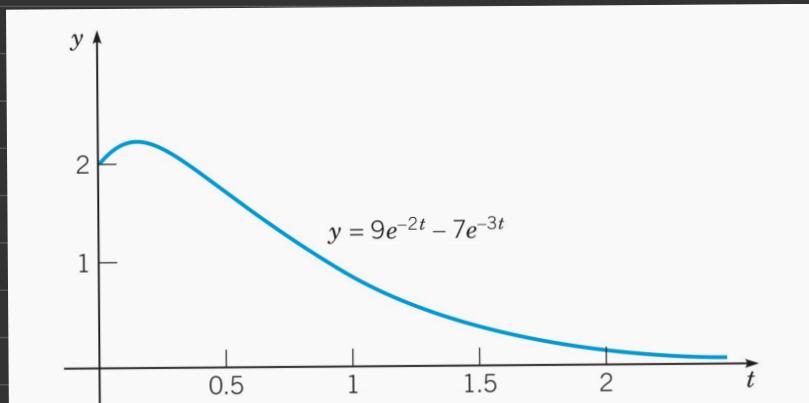
$$7 = -1C_1 \Rightarrow C_1 = -7$$

$$C_2 \cdot 2 = -7 + C_2$$

$$9 = C_2$$

Soln to IVP :

$$y = -7e^{-3t} + 9e^{-2t}$$



**FIGURE 3.1.1** Solution of the initial value problem (28):  
 $y'' + 5y' + 6y = 0, y(0) = 2, y'(0) = 3.$

In Summary : Given  $ay'' + by' + cy = 0$ .

Then Char. poly :  $ar^2 + br + c = 0$

assuming the solution form is  $y = e^{rt}$

④ Real Distinct roots to the Char. poly.

$$\text{Soln} : y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Given I.C :  $y(t_0) = y_0, y'(t_0) = y_1$

## 3.2 The Wronskian ; linearity of Solutions

Cramer : Given  $a_1x + b_1y = c_1$

$$a_2x + b_2y = c_2$$

Solve for  $x$  and  $y$ .

The determinant of this system is

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$$

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - b_1c_2$$

$$x = \boxed{\frac{D_x}{D}} \quad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - c_1a_2$$

$$\boxed{y = \frac{D_y}{D}}$$

Next time : Given  $y'' + p(t)y' + q(t)y = 0$

Solution :  $y = C_1y_1 + C_2y_2$ .

*Big Question: What is  $C_1$  and  $C_2$ ?*

*To be continued...*