

# Math 39100 : Mar. 15. 2023 : LECTURE 13

Quiz 3 : Monday, Mar 20 (Section 3.5).

## § 3.6 cont.

Cont. from last time :

$$u = -4 \left( t - \frac{\sin(2t)}{2} \right) + C_1$$
$$= \boxed{-4t + 2\sin(2t) + C_1}$$

$$V = \int \frac{gy_1}{aw} dt = \int \frac{8 \cdot \tan(t) \cdot \cos(2t)}{2} dt$$
$$= 4 \int \tan(t) \cdot (2\cos^2 t - 1) dt$$
$$= 4 \int 2 \cdot \frac{\sin(t)}{\cos(t)} \cdot \cos^2(t) dt$$
$$= 4 \int \tan(t) dt$$

$$= 8 \int \sin(t) \cos(t) dt + 4(\ln|\cos t|)$$

useful identities

$$(i) \sin(2x) = 2\sin x \cos x$$

$$(ii) \cos(2x) = \boxed{\cos^2 x - \sin^2 x}$$

$$= \cos^2 x - (1 - \cos^2 x)$$

$$= \boxed{2\cos^2 x - 1}$$

or  $= \boxed{1 - 2\sin^2 x}$

$$\text{let } u = \sin(t)$$

$$du = \cos(t) dt$$

$$= 8 \int u du + 4 \ln|\cos(t)|$$

$$= 4u^2 + 4 \ln|\cos(t)| + C_1$$

$$= \boxed{4 \sin^2(t) + 4 \ln|\cos(t)| + C_1}$$

$$\text{Gen. Soln: } y = u y_1 + v y_2$$

$$= (-4t + 2 \sin(2t) + C_1) \cos(2t)$$

$$+ (4 \sin^2(t) + 4 \ln|\cos(t)| + C_2) \sin(2t)$$

$$= -4t \cos(2t) + 2 \sin(2t) \cos(2t) + 4 \sin^2(t) \sin(2t)$$

$$+ 4 \ln|\cos(t)| \cdot \sin(2t)$$

$$\underbrace{\quad}_{y_p}$$

$$+ C_1 \cos(2t) + C_2 \sin(2t)$$

$$\underbrace{\quad}_{y_h}$$

$$\text{eg. Solve } y'' + 4y' + 4y = \underbrace{t^2 e^{-2t}}_{g(t)}, t > 0.$$

$$y_h: \quad y'' + 4y' + 4y = 0$$

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \Rightarrow r_1 = r_2 = -2$$

$$y_h = C_1 \underbrace{e^{-2t}}_{y_1} + C_2 t \underbrace{e^{-2t}}_{y_2}$$

$$W: \quad W = \begin{vmatrix} e^{-2t} & t e^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix}$$

$$= e^{-4t} - 2te^{-4t} + 2te^{-4t}$$

$$= \boxed{e^{-4t}}$$

Find  $u$  and  $v$ :

$$u = - \int \frac{g y_2}{aw} dt = - \int \frac{\cancel{t^2} e^{-2t} \cdot \cancel{t} e^{-2t}}{\cancel{e^{-4t}}} dt$$

$$= - \int \frac{1}{t} dt$$

$$u = -\ln(t) + C_1$$

$$V = \int \frac{gy_1}{aw} dt = \int \frac{\bar{t}^2 \cdot \frac{-2t}{e^{-4t}} \cdot \bar{e}^{2t}}{e^{-4t}} dt$$

$$= \int \bar{t}^2 dt$$

$$= -\bar{t}^3 + C_2 = \boxed{-\frac{1}{t} + C_2}$$

$$y = uy_1 + vy_2$$

$$= (-\ln(t) + C_1) \bar{e}^{2t} + \left(-\frac{1}{t} + C_2\right) t \bar{e}^{2t}$$

$$y = -\ln(t) \bar{e}^{2t} - \underbrace{\bar{e}^{2t}}_{\substack{e^{-2t} \\ C}} + C_1 \bar{e}^{-2t} + C_2 t \bar{e}^{-2t}$$

$$y = \underbrace{-\ln(t) \bar{e}^{2t}}_{y_p} + \underbrace{(C_1 \bar{e}^{-2t} + C_2 t \bar{e}^{-2t})}_{y_h}$$

## Useful Identities:

Double & : i)  $\sin(2x) = 2\sin(x)\cos(x)$

ii)  $\cos(2x) = \cos^2 x - \sin^2 x$

iii)  $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

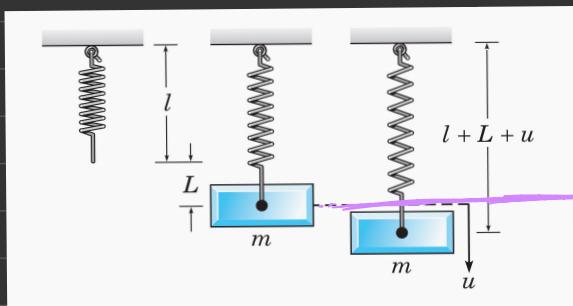
iv)  $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

## § 3.7 / 3.8 Applications of 2<sup>nd</sup> order D.E (Spring-mass problems).

We want to model:

$$ay'' + by' + cy = f(t), \quad y(0) =$$

$$y'(0) =$$



$l$  = natural length of  
the spring

Equilibrium position

$L$  = length spring is

stretched

let  $u(t)$  = position of the mass (position function)

$u'(t)$  = velocity function

$u''(t)$  = acceleration function.



$$F_g = \text{gravitational force} \Rightarrow W = Mg \quad (\text{Hooke's})$$

$$F_s = \text{spring force} \Rightarrow F_s = -kL, k = \text{spring constant}$$

$\alpha F_s = -k(\underbrace{L+u}_{\text{total elongation}})$

$$\textcircled{1} \quad Mg - kL = 0$$

$$F_d = \text{damping force} \Rightarrow F_d = \boxed{\frac{\text{resistance force}}{\text{Velocity}} \cdot u'} \quad \text{called } \gamma$$

So,  $F_d = -ru$

By Newton's 2<sup>nd</sup> law :  $\sum F = ma$

$$\sum F = mu''$$

$$mg + -k(L+u) - \gamma u' + F(t) = mu''$$

$$\underbrace{mg - kL - ku - \gamma u'}_0 + F(t) = mu''$$

$$-ku - \gamma u' + F(t) = mu''$$

$$\Rightarrow \boxed{mu'' + ru' + ku = F(t)}$$



Describes the motion of the mass.

I.C.



$u(0) = \text{initial position of the mass}$

$u'(0) = \text{initial velocity of the mass}$

How to Find  $m, k, r$  ?

$$m : W = mg \Rightarrow m = \frac{W}{g}$$

use

$$g = 9.8 \text{ m/sec}^2$$

or  $32 \text{ ft/sec}^2$

$$k : W = KL \Rightarrow k = \frac{W}{L}$$

$$r : r = \frac{\text{resistance force}}{\text{velocity}}$$

Common units:

$$\text{in} \rightarrow \text{ft}$$

$$\text{cm} \rightarrow \text{m}$$

$$\gamma \rightarrow k_g$$

W

L

A mass weighing 4 lb stretches a spring 2 in. Suppose that the mass is given an additional 6-in displacement in the positive direction and then released. The mass is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/s. Under the assumptions discussed in this section, formulate the initial value problem that governs the motion of the mass.

$$M\ddot{u}'' + D\dot{u}' + Ku = F(t), \quad u(0) = 6 \text{ in.} \cdot \frac{1 \text{ ft}}{12 \text{ in.}} = \frac{1}{2} \text{ ft}$$

$$u'(0) = 0 \text{ (released)}$$

$$W = 4 \text{ lb} \implies M = \frac{W}{g}$$

$$(g = 32 \text{ ft/sec}^2)$$

$$(M) = \frac{4}{32} = \left(\frac{1}{8}\right)$$

$$(L) = 2 \text{ in.} \cdot \frac{1 \text{ ft}}{12 \text{ in.}} = \left(\frac{1}{6} \text{ ft}\right) \implies (K) = \frac{W}{L} = \frac{4}{\frac{1}{6}} = (24)$$

$$F: \frac{\text{resist. force}}{\text{velocity}} = \frac{6 \text{ lb}}{3 \text{ ft/s}} = (2 \text{ lb} \cdot \text{s} \cdot \text{ft})$$

$$\frac{1}{8} \ddot{u}'' + 2\dot{u}' + 24u = 0$$

Since external force is not mentioned.

$$u(0) = \frac{1}{2} \text{ ft}$$

$$u'(0) = 0$$