

Math 39100 : April.26. 2023 : LECTURE 22

Exam 2: Monday, May 1 (starting from 3.5, ...)

Exam 3: (optional) Friday, May 12

Time: 4-5:15 pm

Room: Mac 1/511

FINAL EXAM: May, 18 from 3:30 - 5:45 pm

Room: TBA

6.2 cont. from last time:

$$y = \mathcal{L}^{-1} \left\{ \frac{113}{s-2} \right\} + \mathcal{L}^{-1} \left\{ \frac{2/3}{s+1} \right\}$$

$$= \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} + \frac{2}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\}$$

[using the table]

$$y = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$$

Find the solution of the differential equation

$$y'' + y = \sin(2t)$$

satisfying the initial conditions

$$y(0) = 2, \quad y'(0) = 1.$$

⊗ Find the Laplace Transform of



$$\mathcal{L}\{y''\} = s^2 \mathcal{L}\{y\} - sy(0) - y'(0)$$

$$\mathcal{L}\{y'\} = s \mathcal{L}\{y\} - y(0)$$

$$\int_0^{\infty} e^{-st} \sin(2t) dt$$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin(2t)\}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \frac{2}{s^2 + 4}$$

$$s^2 \mathcal{L}\{y\} - \underbrace{sy(0)}_2 - \underbrace{y'(0)}_1 + \mathcal{L}\{y\} = \frac{2}{s^2 + 4}$$

$$\text{let } Y(s) = \mathcal{L}\{y\}$$

$$s^2 Y(s) - 2s - 1 + Y(s) = \frac{2}{s^2 + 4}$$

$$Y(s)(s^2 + 1) = \frac{2}{s^2 + 4} + 2s + 1$$

$$Y(s) = \frac{2 + 2s^3 + 8s + s^2 + 4}{(s^2 + 1)(s^2 + 4)}$$

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}$$

$$Y(s) = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4}$$

$$\frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

⋮
finish me.

Solve the initial value problem by using the Laplace transform:

$$y'' - 4y' + 4y = 3, \quad y(0) = 0, \quad y'(0) = 1$$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{3\}$$

$$\text{let } \mathcal{L}\{y\} = Y(s)$$

$$s^2 Y(s) - s y(0) - y'(0) - 4(s Y(s) - y(0)) + 4Y(s) = \frac{3}{s}$$

$$s^2 Y(s) - s \underbrace{y(0)}_0 - \underbrace{y'(0)}_1 - 4s Y(s) + 4 \underbrace{y(0)}_0 + 4Y(s) = \frac{3}{s}$$

$$s^2 Y(s) - 1 - 4s Y(s) + 4Y(s) = \frac{3}{s}$$

$$Y(s) (s^2 - 4s + 4) = \frac{3 + s}{s}$$

$$Y(s) = \frac{3 + s}{s(s^2 - 4s + 4)} = \frac{s + 3}{s(s-2)^2}$$

$$= \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$s+3 = A(s-2)^2 + Bs(s-2) + Cs$$

$$s=2: \quad s=2c \Rightarrow C = \frac{5}{2}$$

$$s=0: \quad 3 = A(-2)^2 \Rightarrow A = \frac{3}{4}$$

$$s=1: \quad B = 4 - \frac{3}{4} - \frac{5}{2} = \frac{16-3-10}{4}$$

$$B = \frac{3}{4}$$

$$Y(s) = \frac{3/4}{s} + \frac{3/4}{s-2} + \frac{5/2}{(s-2)^2}$$

to find y : we apply inverse L.T:

$$y = \mathcal{L}^{-1} \left\{ \frac{3/4}{s} \right\} + \frac{3}{4} \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} + \frac{5}{2} \mathcal{L}^{-1} \left\{ \frac{1!}{(s-2)^{1+1}} \right\}$$

$$y = \frac{3}{4} + \frac{3}{4} e^{2t} + \frac{5}{2} t e^{2t}$$

(1) on
the
table

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

Exam 2: (3.5, 3.6, 3.7 (3.8), 4.2, 4.3, 5.1, 5.2, 5.3, 5.4,
5.5, 6.1 and 6.2.)

about 8-10 questions; multiple choice and short response.

• Solving S.O.C. D.E. Nonhomogeneous — Unde. Coeff.
— Variation of parameters

$$y = uy_1 + vy_2 \quad \text{where}$$

$$u = - \int \frac{gy_2}{aw} dt \quad \text{and}$$

$$v = \int \frac{gy_1}{aw} dt$$

• Higher Order Homog. and Nonhomogeneous

• Spring-mass : $mz'' + \gamma z' + kz = G(t)$, $z(0)$
 $z'(0)$

$$m = \frac{W}{g} \quad \text{and} \quad \gamma = \frac{\text{resist. force}}{\text{velocity}}$$

$$k = \frac{W}{L}$$

s.1: Ratio Test to Find I.O.C and R.O.C.

Series Solutions: i) near ordinary point

ii) Regular Singular point

Singular points: Given $P(x)y'' + Q(x)y' + R(x)y = 0$

$x_0 = \text{singular point}$: (1) $P(x_0) = 0$

$$(2) \lim_{x \rightarrow x_0} (x-x_0) \frac{Q}{P} = \text{Finite} \quad \left. \vphantom{\lim} \right\} \Rightarrow x_0 \text{ is R.S.P.}$$

$$(3) \lim_{x \rightarrow x_0} (x-x_0)^2 \frac{R}{P} = \text{finite}$$

if (2) or (3) fails, then x_0 is irregular S.P.

Solving Euler Equations: $\alpha x^2 y'' + \beta x y' + \gamma y = 0$

Def. of Laplace Transform.

Using L.T to solve S.O.L.D.E.

Find power series solution near x_0 .

$$y'' - xy' - y = 0, \quad x_0 = 1$$

$$y = \sum a_n (x-1)^n = a_0 + a_1(x-1) + a_2(x-1)^2 + \dots$$

$$y' = \sum n a_n (x-1)^{n-1}$$

$$y'' = \sum n(n-1) a_n (x-1)^{n-2}$$

$$\sum n(n-1) a_n (x-1)^{n-2} - x \sum n a_n (x-1)^{n-1} - \sum a_n (x-1)^n = 0$$

$$\text{let } z = x-1 \Rightarrow x = z+1$$

$$\sum n(n-1) a_n z^{n-2} - (z+1) \sum n a_n z^{n-1} - \sum a_n z^n = 0$$

$$\sum n(n-1) a_n z^{n-2} - z \sum n a_n z^{n-1} + \sum n a_n z^{n-1} - \sum a_n z^n = 0$$

⋮

$$y = a_0 y_1 + a_1 y_2$$

Additional practice problems:

1. Using the definition, compute the Laplace Transform of the function $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 1, & 1 \leq t \leq \infty \end{cases}$.

I. $F(s) = \frac{1}{s^2}, s > 0$

II. $F(s) = \frac{1}{s^2}(1 - e^{-s}) - \frac{2}{s}e^{-s}, s > 0$

III. $F(s) = \frac{1}{s^2}(1 - e^{-s}), s > 0$

IV. $F(s) = \frac{1}{s^3}, s > 0$

V. $F(s) = \frac{1}{s^2}(1 + e^{-s}), s > 0$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^{\infty} e^{-st} \cdot f(t) dt \\ &= \int_0^1 e^{-st} \cdot t dt + \int_1^{\infty} e^{-st} \cdot 1 dt \end{aligned}$$

2. For the given differential equation:

$$x^2 y'' + xy' - 3xy = 0$$

We seek a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Using the larger root, the recursion formula is given by

3. For the given differential equation: $(3x^3 + x^2)y'' + xy' + (\frac{1}{2} - 5x)y = 0$

Note that $x_0 = 0$ is a Regular Singular Point. Compute the associated Euler equation.

I. $3x^2 y'' + xy' + y = 0$

II. $x^2 y'' + xy' - 5y = 0$

III. $2x^2 y'' + 2xy' + y = 0$

IV. $2x^2 y'' + xy' + y = 0$

5. A 16 pound weight stretches a spring 6 inches. Assuming no friction and no external force, the natural frequency ω for the system is ? (Recall that the acceleration due to gravity is 32 ft/sec^2).

Find the form of y_p .

$$y^{(5)} + 2y^{(4)} + 5y''' + 10y'' = \cos(\sqrt{5}x) - \sin(5x) - 6 - x + xe^{2x}$$

Compute the general solution of $y'' + 6y' + 9y = t^{-2}e^{-3t}$

Compute the general solution of $x^2 y'' + 3xy' - 35y = 0$ with $x > 0$.

17. Use the method of Reduction of Order to find a second, linearly independent solution $y_2(x)$, given that $y_1(x) = x$ is a solution to the differential equation:

$$3x^2 y'' + 2xy' - 2y = 0, x > 0$$

$$y = v y_1 = vx \dots$$