

Math 39100 : Mar. 8 . 2023 : LECTURE 11

§ 3.5 Nonhomogeneous D.E : Method of Undetermined

Coefficients (Specific Method)

Suppose $y'' + p(t)y' + q(t)y = g(t)$, where $p(t)$, $q(t)$ and $g(t)$ are continuous on some open interval I , and $g(t) \neq 0$. (Nonhomogeneous).

$$g(t) = \begin{cases} \text{polynomial} : a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, & n \geq 0 \\ e^{\alpha x} \\ \sin(\alpha x) \text{ or } \cos(\alpha x) \end{cases}, \quad \text{integers}$$

Steps for Undet. Coeff method:

- 1) Find the solution the corresponding homogeneous equation: y_h
- 2) Guess a form for the particular solution: y_p [text: y_c]
[based on the given $g(t)$]
- 3) If there is any duplication in the assumption of y_p appears in y_h , then we multiply y_p by t or t^2 , ...

until its gone.

4) Compute y_p' , y_p'' and plug into the given D.E. to find the coeff.

5) The general is $y = y_h + y_p$.

Proof of 5: Given $y'' + p(t)y' + q(t)y = g(t)$.

Assume that $y = y_h + y_p$ is its general soln.

Note that $y - y_p = y_h$.

$$\begin{aligned} \text{So then, } & (y - y_p)'' + p(t)(y - y_p)' + q(t)(y - y_p) \\ = & y'' - y_p'' + p(t)y' - p(t)y_p' + q(t)y - q(t)y_p \\ = & y'' + p(t)y' + q(t)y - (y_p'' + p y_p' + q y_p) \\ = & g(t) - g(t) \end{aligned}$$

$$= 0$$

$$\Rightarrow y = y_h + y_p.$$

□

Theorem 3.5.2

$$y'' + ay' + by = g(t)$$

The general solution of the nonhomogeneous equation (1) can be written in the form

$$y = \phi(t) = \underbrace{c_1 y_1(t) + c_2 y_2(t)}_{y_h} + \underbrace{Y(t)}_{y_p}, \quad (7)$$

where y_1 and y_2 form a fundamental set of solutions of the corresponding homogeneous equation (2), c_1 and c_2 are arbitrary constants, and Y_p is any solution of the nonhomogeneous equation (1).

1. Find the general solution $c_1 y_1(t) + c_2 y_2(t)$ of the corresponding homogeneous equation. This solution is frequently called the **complementary solution** and may be denoted by $y_c(t) = y_h$.
2. Find any solution $Y(t)$ of the nonhomogeneous equation. Often this solution is referred to as a **particular solution**.
3. Form the sum of the functions found in steps 1 and 2.

Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t}.$$

Step 1: Corresponding homog. eq: $y'' - 3y' - 4y = 0$.

$$\text{char. poly: } r^2 - 3r - 4 = 0$$

$$(r + 1)(r - 4) = 0$$

$$r_1 = -1, r_2 = 4$$

$$y_h = C_1 e^{-t} + C_2 e^{4t}$$

Step 2 & 3: $g(t) = 3e^{2t}$

Guess: $y_p = A e^{2t}$ ← no need to modify since it does

not appear in y_h .

Step 4: $y_p' = 2Ae^{2t}$

$$y_p'' = 4Ae^{2t}$$

$$y'' - 3y' - 4y = 3e^{2t}.$$

plus in y_p, y_p', y_p'' :

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$$

$$e^{2t}(4A - 6A - 4A) = 3e^{2t}$$

$$-6Ae^{2t} = 3e^{2t}$$

$$-6A = 3 \Rightarrow A = -\frac{1}{2}$$

$$y_p = -\frac{1}{2}e^{2t}$$

particular solution.

Step 5:

General solution: $y = y_h + y_p$

$$\Rightarrow y = C_1 e^{-t} + C_2 e^{4t} - \frac{1}{2}e^{2t}$$

Q1. Given $y_h = C_1 + C_2 e^t$ and $g(t) = -3$.

$$y_p = At$$

Q2. Given $y_h = C_1 e^{-4t} + C_2 t e^{-4t}$ and $g(t) = \underbrace{q}_{y_p} + \underbrace{e^{-4t}}_{y_p}$

$$y_p = y_{p_1} + y_{p_2}$$

$$y_p = A + B e^{-4t} \cdot t^2$$

Q3. Given $y_h = C_1 \cos(t) + C_2 \sin(t)$ and $g(t) = 3t \sin(t)$.

$$y_p = [(A t + B) \sin(t) + (C t + D) \cos(t)] t$$

How to guess y_p :

$g(t)$

1) polynomial of degree n

$$\text{eg. } 3t^2 - 4$$

2) poly and exponential

3) poly, exponential and sine
or cosine

y_p to be...

$$t^s (a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t^2 + a_0 t + a_0)$$

$s = 0, 1, 2, 3, \dots$

$$At^2 + Bt + C$$

$$t^s [(a_n t^n + a_{n-1} t^{n-1} + \dots + a_0) e^{\alpha t}]$$

$$t^s [(a_n t^n + a_{n-1} t^{n-1} + \dots + a_0) e^{\alpha t} \cdot \cos(\beta t) + (b_n t^n + b_{n-1} t^{n-1} + \dots + b_0) e^{\alpha t} \sin(\beta t)]$$

above t^s , where $s = 0, 1, 2, 3, \dots$ this will ensure any duplication from y_h is removed from y_p .

e.g. Find the general solution:

$$y'' - y' - 2y = -2t + 4t^2$$

$$y'' - y' - 2y = 0$$

$$\Rightarrow r^2 - r - 2 = 0$$

$$(r - 2)(r + 1) = 0$$

$$r_1 = 2, r_2 = -1$$

$$y_h = C_1 e^{2t} + C_2 e^{-t}$$

$$y_p = At^2 + Bt + C$$

$$y_p' = 2At + B$$

$$y_p'' = 2A$$

plug in y_p , y_p' and y_p'' :

$$2A - (2At + B) - 2(At^2 + Bt + C) = -2t + 4t^2$$

$$2A - 2At - B - 2At^2 - 2Bt - 2C = -2t + 4t^2$$

$$-2A - 2B = -2 \quad \rightarrow \quad 4 - 2B = -2$$

$$-2A = 4 \Rightarrow A = -2$$

$$-2B = -6$$

$$B = 3$$

$$2A - B - 2C = 0$$

:

$$C = -\frac{7}{2}$$

$$y_p = -2t^2 + 3t - \frac{7}{2}$$

particular soln.

Gener. soln: $y = y_h + y_p$

$$y = C_1 e^{2t} + C_2 e^{-t}$$

$$-2t^2 + 3t - \frac{7}{2}$$

eg. Solve the IVP: $y'' + 4y = 3\sin(t)$,

$$y(0) = 2, \text{ and } y'(0) = -1.$$

$$y_h: y'' + 4y = 0$$

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

$$y_h = C_1 \cos(2t) + C_2 \sin(2t)$$

$$y_p : \boxed{y_p = A \sin(t) + B \cos(t)} \quad \leftarrow \text{no need to modify.}$$

$$y_p' = A \cos(t) - B \sin(t)$$

$$y_p'' = -A \sin(t) - B \cos(t)$$

plug in y_p , y_p' and y_p'' into original eq:

$$-A \sin(t) - B \cos(t) + 4(A \sin(t) + B \cos(t)) = 3 \sin(t)$$

$$-A + 4A = 3$$

$$\boxed{A = 1}$$

$$-B + 4B = 0$$

$$\boxed{B = 0}$$

$$\boxed{y_p = \sin(t)}$$

$$\underline{\text{Gen. Soln:}} \quad y = y_h + y_p$$

$$y = C_1 \cos(2t) + C_2 \sin(2t) + \sin(t)$$

$$y(0) = 2 \Rightarrow$$

$$y'(0) = -1 \Rightarrow \dots \text{to be continued...}$$