Math 39100 : Feb. 22. 2023 : LECTURE 8

Exam 1: Monday, March 6

Warm up: Solve $2y'' + 7y' - 4y = 0$.

Char poly: $2r^2 + 7r - 4 = 0$

$$(2r-1)(r+4) = 0$$

$$r_1 = \tfrac{1}{2}, \quad r_2 = -4$$

$$y_1 = e^{\frac{1}{2}t}, \quad y_2 = e^{-4t}$$

$$\boxed{y = C_1 e^{\frac{1}{2}t} + C_2 e^{-4t}}$$ family of solutions.

§ 3.2 CONT.

Suppose we have IVP: $y'' + p(t)y' + q(t)y = 0$,

$y(t_0) = y_0$ and $y'(t_0) = y_0'$.

The solution: $y = C_1 y_1 + C_2 y_2$.

$$y' = C_1 y_1' + C_2 y_2'$$

$$y(t_0) = y_0 \Rightarrow \begin{cases} \overset{\text{"x"}}{C_1}\, y_1(t_0) + \overset{\text{"y"}}{C_2}\, y_2(t_0) = y_0 \end{cases}$$

$y'(t_0) = y_0' \Rightarrow \left\{ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \right.$

Determinate : $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

This is known as The Wronskian

Def:

Wronkian of two solutions y_1, y_2 : $W(y_1, y_2) = W$

$$\boxed{W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'}$$

IF $W \neq 0$, then (i) Find unique values for c_1 and c_2.

(ii) solutions y_1 and y_2 are called fundamental set of solutions.

(iii) y_1 and y_2 are linearly independent.

Find values for c_1 and c_2 :

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2 \\ y_0' & y_2' \end{vmatrix}}{W} = \frac{y_0 y_2' - y_2 y_0'}{W}$$

$$C_2 = \frac{\begin{vmatrix} y_1 & y_0 \\ y_1' & y_0' \end{vmatrix}}{W} = \frac{y_1 y_0' - y_0 y_1'}{W}$$

Theorem 3.2.3

Suppose that y_1 and y_2 are two solutions of equation (2)

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

L = differential operator

and that the initial conditions (3)

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

are assigned. Then it is always possible to choose the constants c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation (2) and the initial conditions (3) if and only if the Wronskian

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2$$

is not zero at t_0.

we found that $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$ are solutions of

$$y'' + 5y' + 6y = 0.$$

Find the Wronskian of y_1 and y_2.

$$W(y_1, y_2) = W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$+ \qquad -$

$$= \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -3e^{-2t} \cdot e^{-3t} + 2e^{-2t} \cdot e^{-3t}$$

$$= -3e^{-5t} + 2e^{-5t}$$

$$= \boxed{-e^{-st}}$$

Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0.$$

$y_1 = t^{1/2}$ $\qquad$ $y_2 = t^{-1}$

$y_1' = \frac{1}{2}t^{-1/2}$ $\qquad$ $y_2' = -t^{-2}$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix}$$

$$= -t^{1/2} \cdot t^{-2} - \frac{1}{2}t^{-1} \cdot t^{-1/2}$$

$$= -t^{-3/2} - \frac{1}{2}t^{-3/2}$$

$$= -\frac{3}{2}t^{-3/2} \neq 0$$

Thus, y_1 and y_2 form a fundamental set of solution.

Theorem 3.2.1 | (Existence and Uniqueness Theorem)

Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{4}$$

where p, q, and g are continuous on an open interval I that contains the point t_0. This problem has exactly one solution $y = \phi(t)$, and the solution exists throughout the interval I.

Theorem 3.2.2 | (Principle of Superposition)

If y_1 and y_2 are two solutions of the differential equation (2),

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0,$$

then the linear combination $c_1 y_1 + c_2 y_2$ is also a solution for any values of the constants c_1 and c_2.

proof: Assume y_1 and y_2 are solutions to

$$L[y] = y'' + p(t)y' + q(t)y = 0. \quad (1)$$

Then $L[y_1] = y_1'' + p(t)y_1' + q(t)y_1 = 0$ and

$L[y_2] = y_2'' + p(t)y_2' + q(t)y_2 = 0.$

We want to show that $L[c_1y_1 + c_2y_2]$ solves the eq.(1).

$$L[c_1y_1 + c_2y_2] = [c_1y_1 + c_2y_2]'' + p(t)[c_1y_1 + c_2y_2]' + q(t)[c_1y_1 + c_2y_2]$$

$$= c_1y_1'' + c_2y_2'' + p(t)c_1y_1' + p(t)c_2y_2' + q(t)c_1y_1 + q(t)c_2y_2$$

$$= c_1\underbrace{[y_1'' + p(t)y_1' + q(t)y_1]}_{L[y_1]} + c_2\underbrace{[y_2'' + p(t)y_2' + q(t)y_2]}_{L[y_2]}$$

$$= c_1 L[y_1] + c_2 L[y_2]$$

Since $L[y_1] = 0$ and $L[y_2] = 0$

Thus, we have $L[c_1 y_1 + c_2 y_2] = 0$.

$\square$

Theorem 3.2.7 | (Abel's Theorem)[4]

If y_1 and y_2 are solutions of the second-order linear differential equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0, \tag{22}$$

where p and q are continuous on an open interval I, then the Wronskian $W[y_1, y_2](t)$ is given by

$$W[y_1, y_2](t) = c\exp\left(-\int p(t)\,dt\right), = c\,e^{-\int p(t)\,dt} \tag{23}$$

where c is a certain constant that depends on y_1 and y_2, but not on t. Further, $W[y_1, y_2](t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

<u>proof:</u> y_1 and y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$.

Then, $y_1'' + p(t)y_1' + q(t)y_1 = 0$ (eq1) and

$y_2'' + p(t)y_2' + q(t)y_2 = 0$. (eq2)

Multiply (eq1) by $-y_2$ and (eq2) by y_1. Then

$$-y_1''y_2 - p(t)y_1'y_2 - \cancel{q(t)y_1y_2} = 0$$

$(+)$

$$y_1y_2'' + p(t)y_1y_2' + \cancel{q(t)y_1y_2} = 0$$

$$\underbrace{y_1 y_2'' - y_1'' y_2}_{W'} + p\underbrace{(y_1 y_2' - y_1' y_2)}_{W} = 0$$

$$W' + pW = 0$$

$$\frac{dW}{dt} + pW = 0 \quad \longleftarrow \text{Separable \& Linear}$$

$$\int \frac{dW}{W} = -\int p \, dt$$

$$\ln|W| = -\int p \, dt + C$$

$W > 0$, $$W = Ce^{-\int p(t)\,dt}.$$

$W \neq 0$ unless $C = 0$. $\square$

eg. Given $y'' - 4y' + 3y = 0$. Use Abel's theorem to compute the wronskian of the solutions.

$p(t) = -4$, $$W = ce^{-\int -4\,dt} = \boxed{ce^{4t}}$$