

Math 39100 : May. 10 . 2023 : LECTURE 25

Exam 3: (optional) Friday, May 12

Time: 4 - 5:15 pm

Room: Nac 11511

Topics: CH2 (Solving 1st order eq.) , CH3 (Solving 2nd order eq.)

5.2, 10.2, 10.4 and 10.5.

Quiz 6: Monday, May 15 : on sections 10.2/10.4 and 10.5

FINAL EXAM: May, 18 from 3:30 - 5:45 pm

Room: Nac 01201

§ 10.5 Separation of Variables ; Heat Condition

in a Rod

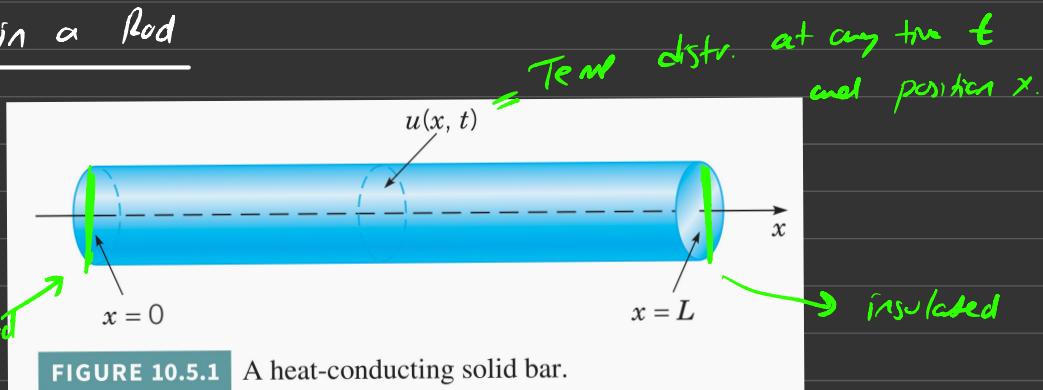


FIGURE 10.5.1 A heat-conducting solid bar.

$$\text{PDE : } \begin{cases} \alpha^2 u_{xx} = u_t \\ u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0 \quad (\text{B.C.}) \\ u(x, 0) = f(x) \quad \leftarrow \text{I.C.} \end{cases}$$

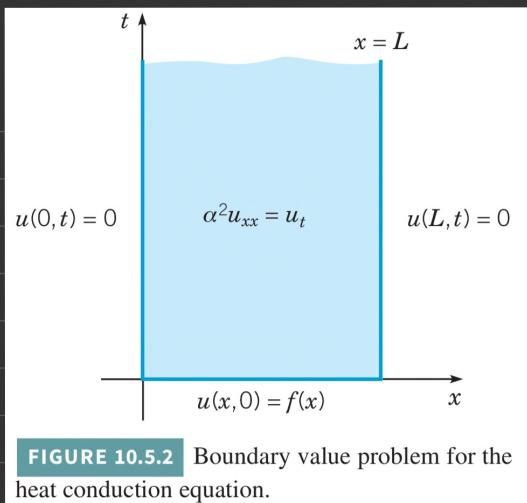


FIGURE 10.5.2 Boundary value problem for the heat conduction equation.

Separation of Variables:

$$u_t = \alpha^2 u_{xx}$$

$$\text{Assume } u(x, t) = X(x)T(t) = XT$$

$$u_t = T'X$$

$$u_{xx} = X''T$$

$$\text{So, } T'X = \alpha^2 X''T$$

$$\text{Then } \frac{X''}{X} = \frac{1}{\alpha^2} \frac{T'}{T} = \lambda$$

$$\textcircled{1} \quad \frac{X''}{X} = \lambda \Rightarrow \boxed{X'' - \lambda X = 0}$$

$$\textcircled{2} \quad \frac{T'}{\alpha^2 T} = \lambda \Rightarrow \boxed{T' - \lambda \alpha^2 T = 0}$$

Three cases for λ : (i) $\lambda = 0$ (ii) $\lambda > 0$ (iii) $\lambda < 0$

(i) $\lambda = 0 \Rightarrow X'' = 0$

$$X(x) = Ax + B$$

Applying B.C. $X(0) = 0 \Rightarrow B = 0$

$$X(L) = 0 \Rightarrow AL = 0 \Rightarrow A = 0$$

Soln: $X(x) = 0$ (not eigen soln)

(ii) $\lambda > 0 \Rightarrow \lambda = \lambda^2$

$$X'' - \lambda^2 x = 0$$

$$r^2 - \lambda^2 = 0 \quad (\text{char})$$

$$r = \pm \lambda$$

$$X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

B.C: $X(0) = 0 \Rightarrow 0 = C_1 + C_2 \Rightarrow C_1 = -C_2$

$$X(L) = 0 \Rightarrow 0 = C_1 e^{\lambda L} + C_2 e^{-\lambda L}$$

$$C_1 = 0 \text{ or } C_2 = 0$$

again, No Eigen solution.

$$(i) \quad \lambda < 0 \quad (\lambda = -\lambda^2)$$

$$x'' + \lambda^2 x = 0$$

$$r^2 + \lambda^2 = 0 \Rightarrow r = \pm \lambda i$$

$$X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$$

$$B.C : \quad X(0) = 0 \Rightarrow \boxed{C_1 = 0}$$

$$X(L) = 0 \Rightarrow C_2 \sin(\lambda L) = 0$$

$$\sin(\lambda L) = 0$$

$$\lambda L = n\pi, \quad n \in \mathbb{Z}^+$$

$$\boxed{\lambda = \frac{n\pi}{L}}$$

$$\boxed{X(x) = C_1 \sin\left(\frac{n\pi}{L}x\right)}$$

$$T' + \lambda^2 \alpha^2 T = 0$$

$$T' = -\lambda^2 \alpha^2 T$$

$$\int \frac{T'}{T} = \int -\lambda^2 \alpha^2 dt$$

$$\ln T = -\lambda^2 \alpha^2 t + C$$

$$T(t) = Ce^{-\lambda^2 \alpha^2 t} \Rightarrow \boxed{T(t) = Ce^{-\frac{\lambda^2 \pi^2 \alpha^2}{L^2} t}}$$

$$SOL: \quad U(x,t) = X(x) T(t)$$

$$= C_1 \sin\left(\frac{n\pi x}{L}\right) \cdot C e^{-\frac{\lambda^2 \pi^2 \alpha^2}{L^2} t}$$

$$U(x,t) = B \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\lambda^2 \pi^2 \alpha^2}{L^2} t}, \quad n \in \mathbb{Z}^+$$

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\lambda^2 \pi^2 \alpha^2}{L^2} t}$$

$$I.C: \quad U(x,0) = f(x) \Rightarrow f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

↗
F.S.S.

$$B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

(10) (a) Compute the sine series for the function $f(x) = 1$ on the interval $[0, \pi]$.

(b) Use your answer to part (a) to obtain a series solution $u(t, x)$ for the partial differential equation with x in the interval $[0, \pi]$ and $t > 0$:

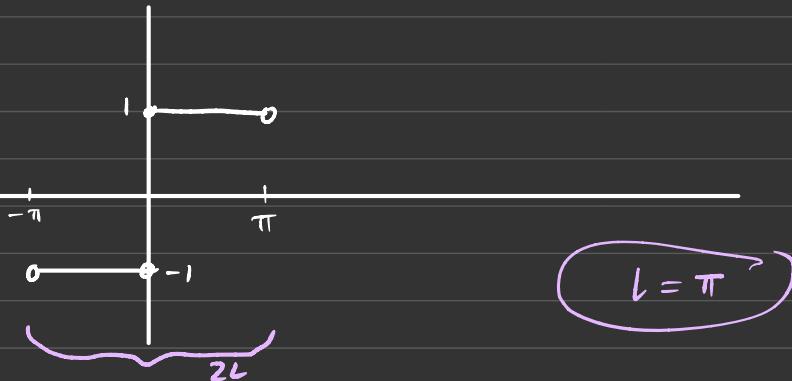
$$u_t = \alpha^2 u_{xx}$$

$$u_t = 64 u_{xx} \quad \text{with}$$

$$u(t, 0) = u(t, \pi) = 0 \quad \text{for } t > 0 \quad (\text{boundary conditions})$$

$$u(0, x) = 1 \quad \text{for } 0 < x < \pi \quad (\text{initial conditions})$$

$= f(x)$



a) $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

$$b_n = \frac{2}{\pi} \int_0^\pi 1 \cdot \sin(nx) dx$$

$$= -\frac{2}{\pi n} \cos(nx) \Big|_0^\pi$$

$$= -\frac{2}{n\pi} \cos(n\pi) + \frac{2}{n\pi} (1)$$

$$= -\frac{2}{n\pi} (-1)^n + \frac{2}{n\pi} = \begin{cases} \frac{4}{n\pi}, & \text{odd} \\ 0, & \text{even} \end{cases}$$

$$\tilde{F}_{\sin e} = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{4}{n\pi} \sin(nx)$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin((2n-1)x)}$$

b) $\boxed{U(x,t)} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}$

$$= \boxed{\sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \cdot \sin((2n-1)x) \cdot e^{-64(2n-1)^2 t}}$$

e.g. Replace the PDE by a pair of ODE's.

$$t U_{xx} + x U_t = 0$$

$$\begin{array}{l} \mathcal{U}(x, t) = X T \\ \mathcal{U}_t = X T' \\ \mathcal{U}_{xx} = X'' T \end{array} \quad \left| \begin{array}{l} t \cdot X'' T + x X \cdot T' = 0 \\ t X'' T = -x X T' \\ \frac{X''}{x X} = -\frac{T'}{t T} = \lambda \end{array} \right.$$

① $\frac{X''}{x X} = \lambda \Rightarrow \boxed{X'' - x X \lambda = 0}$

② $-\frac{T'}{t T} = \lambda \Rightarrow \boxed{T' + \lambda t T = 0}$

The End !