

Math 39100 : Mar. 29. 2023 : LECTURE 17

Quiz 4: Monday, April 3 (on 3.6, and 3.7/3.8)

Exam 2: Monday, May 1 (starting from 3.5, ...)

Exam 3: (optional) Friday, May 12

§ 5.2 cont.

from last time: $y'' + y = 0$

Soln: $y = a_0 \underbrace{\cos(x)}_{y_1} + a_1 \underbrace{\sin(x)}_{y_2}$

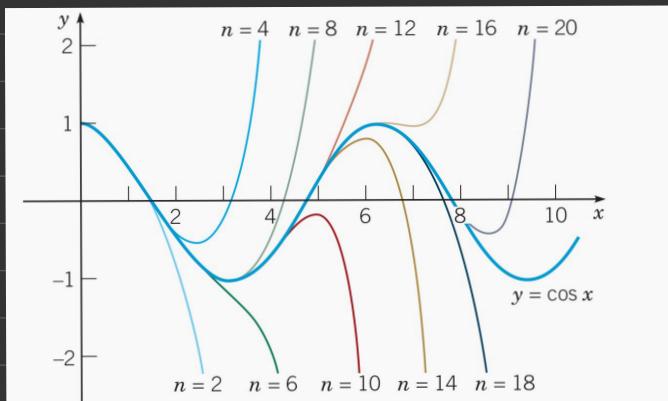


FIGURE 5.2.1 Polynomial approximations to $y = \cos x$. The value of n is the degree of the approximating polynomial.

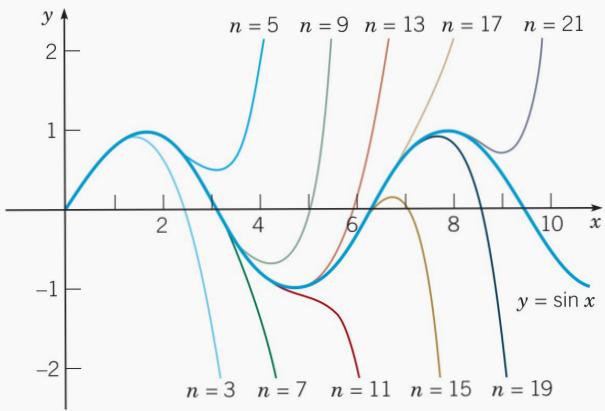


FIGURE 5.2.2 Polynomial approximations to $y = \sin x$. The value of n is the degree of the approximating polynomial.

Find a series solution in powers of x of Airy's⁴ equation

$$y'' - xy = 0, \quad -\infty < x < \infty.$$

Assume the solution form : $y = \sum_{n=0}^{\infty} a_n x^n$, also assume $x_0 = 0$.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

plus in y and y'' into the given DE :

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\begin{cases} k=n-2 \\ n=k+2 \end{cases}$$

$$\begin{cases} k=n+1 \\ n=k-1 \end{cases}$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} a_{k-1} x^k = 0$$

$$2 \cdot 1 \cdot a_2 + \sum_{k=1}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{\infty} a_{k-1} x^k = 0$$

(brace)

$$2a_2 + \sum_{k=1}^{\infty} [(k+2)(k+1) a_{k+2} - a_{k-1}] x^k = 0$$

$$2a_2 = 0 \quad \left| \quad (k+2)(k+1) a_{k+2} - a_{k-1} = 0 \right.$$

$$a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}$$

Recurrence
formula to
get the
coeff.

\textcircled{A} all negative subscript
are zero.

$$k=0, a_2 = 0$$

$$k=1, \boxed{a_3} = \frac{a_0}{2 \cdot 3}$$

$$k=2, \boxed{a_4} = \frac{a_1}{3 \cdot 4} = \frac{a_1}{12}$$

$$k=3, a_5 = \frac{a_2}{5 \cdot 4} \xrightarrow{a_2=0} = 0$$

$$k=4, \quad \boxed{a_6} = \frac{a_3}{6 \cdot 5} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6} = \frac{a_0}{180}$$

Soln: $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \dots$

$$= a_0 + a_1 x + \frac{a_0}{6} x^3 + \frac{a_1}{12} x^4 + \frac{a_0}{180} x^6 + \dots$$

$$\boxed{y = a_0 \left(1 + \frac{1}{6} x^3 + \frac{1}{180} x^6 + \dots \right) + a_1 \left(x + \frac{1}{12} x^4 + \dots \right)}$$

For the differential equation: $(3-x)y'' + y' + y = 0$, compute the recurrence formula for the coefficients of the power series centered at $x_0 = 0$ and use it to compute the first four nonzero terms of the solution with $y(0) = -2$ and $y'(0) = 3$.

$$= a_0 \quad = a_1$$

Steps: Series Solution near ordinary point (assuming $x_0=0$ is an O.P.)

(i) Assume the Soln form: $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

(ii) Find y' and y'' : i.e. $y' = \sum n a_n x^{n-1}$ and

$$y'' = \sum n(n-1) a_n x^{n-2}$$

(ignoring index, see sums).

(iii) plug in y , y' and y'' accordingly in the given D.E. and fix for

power of x to be x^k .

(iv) Combine terms and find a recurrence formula for the coefficients.

(v) Compute a few coeff. in terms of a_0 and a_1 .

(vi) Write the series soln as $y = a_0 y_1 + a_1 y_2$.

$$3y'' - xy'' + y' + y = 0$$

$$3 \sum n(n-1) a_n x^{n-2} - x \sum n(n-1) a_n x^{n-2} + \sum n a_n x^{n-1} + \sum a_n x^n = 0$$

$$\sum 3n(n-1) a_n x^{n-2} - \sum n(n-1) a_n x^{n-1} + \sum n a_n x^{n-1} + \sum a_n x^n = 0$$

$$\begin{cases} k=n-2 \\ n=k+2 \end{cases}$$

$$\begin{cases} k=n-1 \\ n=k+1 \end{cases}$$

$$\begin{cases} k=n-1 \\ n=k+1 \end{cases}$$

$$\begin{cases} k=n \end{cases}$$

$$\sum [3(k+2)(k+1)a_{k+2} - (k+1)ka_{k+1} + (k+1)a_{k+1} + a_k] x^k = 0$$

$$3(k+2)(k+1)a_{k+2} - \underbrace{k(k+1)a_{k+1} + (k+1)a_{k+1} + a_k}_{(-k^2 - k + k+1)a_{k+1}} = 0$$

$$3(k+2)(k+1)a_{k+2} + (1-k^2)a_{k+1} + a_k = 0$$

Recurrence formula

$$a_{k+2} = \frac{-a_k + (k^2 - 1)a_{k+1}}{3(k+2)(k+1)}$$

$$k=0, \quad a_2 = \frac{-a_0 - a_1}{3 \cdot 2 \cdot 1} = -\frac{a_0}{6} - \frac{a_1}{6} \\ = +\frac{2}{6} - \frac{2}{6} = \boxed{-\frac{1}{6}}$$

$$k=1, \quad a_3 = \frac{-a_1 + 0}{3 \cdot 3 \cdot 2} = -\frac{a_1}{18} = \frac{-3}{18} = \boxed{-\frac{1}{6}}$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Initial Condition : $y(0) = -2 \Rightarrow \boxed{-2 = a_0}$

$$y'(0) = 3 \text{ So, } y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$y'(0) = 3 \Rightarrow \boxed{3 = a_1}$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$\Rightarrow \boxed{y = -2 + 3x - \frac{1}{6}x^2 - \frac{1}{6}x^3 + \dots}$$

§ 5.3 Series Soln near ordinary point (Part II)

From Taylor's formula, we know that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{where} \quad \boxed{a_n = \frac{f^{(n)}(x_0)}{n!}}$$

Solve $y'' + \sin(x)y' + \cos(x)y = 0$, $y(0) = a_0$ and
 $y'(0) = a_1$.

Assume $x_0 = 0$.