1. (10 points) The general solution of the differential equation $y' - 2xy = e^{x^2}$ is

a) $y = xe^{x^2} + ce^{x^2}$

b) $y = xe^{-x^2} + ce^{-x^2}$

c) $y = xe^{x^2} + c$

d) $y = xe^{2x} + ce^{2x}$

e) none of the above.

$\mu(x) = e^{\int -2x\,dx} = e^{-x^2}$

$e^{-x^2} \cdot y = \int 1\,dx$

$ye^{-x^2} = x + c$

$y = xe^{x^2} + ce^{x^2}$

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2. (10 points) Compute the general solution of: $(x-1)\cos(y)dy = 2x\sin(y)dx$

a) $\ln|\sin(y)| = \ln|x-1| + C$

b) $\sin(y) = C(x-1)^2 e^{2x}$

c) $\sin(y) = (x-1)e^x + C$

d) $\sin(y)(x-1)^2 = C$

e) none of the above.

$2 \cdot \ln|x-1| \cdot 2x$

$\int \frac{\cos y}{\sin y}\,dy = 2 \cdot \int \frac{x}{x-1}\,dx$ $\quad u = x-1$, $du = 1$

$\ln|\sin y| = 2 \cdot \int \frac{u+1}{u}\,du$ $\quad \frac{u+1}{u}$

$\ln|\sin y| = 2\left(\int du + \int \frac{1}{u}\,du\right)$

$e^{\ln|\sin y|} = e^{2 \cdot u + 2\ln|u| + c}$

$\sin y = e^{2x-2} \cdot (x-1)^2 c$

$\boxed{\sin y = ce^{2x}(x-1)^2}$

3. (10 points) The solution to the IVP $(x - 2xy + e^y)dx + (y - x^2 + xe^y)dy = 0, \quad y(-4) = 0$ is

M N

a) $\frac{1}{2}x^2 + x^2y + xe^y + \frac{1}{2}y^2 = -8$

b) $\frac{1}{2}x^2 - x^2y + e^y + y^2 = 4$

c) $\frac{1}{2}x^2 - x^2y + xe^y + \frac{1}{2}y^2 = 8$

d) $\frac{1}{2}x^2 - x^2y + xe^y + \frac{1}{2}y^2 = 4$

e) none of the above.

$M_y = -2x + e^y \quad N_x = -2x + e^y$ Exact ✓

$\int x - 2xy + e^y \, dx = \frac{x^2}{2} - x^2y + xe^y + g(y)$

$-x^2 + xe^y + g'(y) = y - \cancel{x^2} + \cancel{xe^y}$

$g'(y) = y \quad g(y) = \frac{1}{2}y^2 + c$

$C = \frac{x^2}{2} - x^2y + xe^y + \frac{1}{2}y^2 \quad ; \; y(-4) = 0$

$C = 8 - 0 + (-4) + 0$

$C = 4$ ✓

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4. (10 points) Compute the solution $u(x,t)$ for the partial differential equation with x in the interval $[0,1]$ and $t > 0$:

$$16u_t = u_{xx}$$

with
$$u(0,t) = u(1,t) = 0 \text{ for } t > 0 \text{ (boundary conditions)}$$
$$u(x,0) = \sin(\pi x) - 5\sin(3\pi x) \text{ (initial conditions).}$$

Done in class

5. For the differential equation:

$$xy'' + y' + xy = 0$$

(a) (5 points) Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 1$.

(b) (5 points) Now, use part (a) to compute the first three nonzero terms of the series solutions.

a) $y = \sum_1^\infty a_n (x-1)^n \quad y' = \sum a_n \cdot n (x-1)^{n-1} \quad y'' = \sum a_n n(n-1)(x-1)^{n-2}$

$x \sum_1^\infty a_n n(n-1)(x-1)^{n-2} + \sum a_n \cdot n (x-1)^{n-1} + x \sum_1^\infty a_n (x-1)^n$

$z = x-1$

$(z+1) \sum a_n n(n-1)(z)^{n-2} + \sum a_n \cdot n (z)^{n-1} + (z+1) \sum a_n (z)^n = 0$

$\sum a_n n(n-1) z^{n-1} + \sum a_n n(n-1)(z)^{n-2} + \sum a_n \cdot n (z)^{n-1} + \sum a_n (z)^{n+1} + \sum a_n z^n = 0$

$k = n-1 \quad k = n-2 \quad k = n-1 \quad k = n+1 \quad k = n$

$n = k+1 \quad n = k+2 \quad n = k+1 \quad n = k-1$

$$\sum a_{k+1} (k+1) \cdot k\, z^k + \sum a_{k+2} (k+2)(k+1) z^k + \sum a_{k+1} (k+1) z^k + \sum a_{k-1} z^k + \sum a_k z^k = 0$$

$$\sum \left[k(k+1) a_{k+1} + (k+2)(k+1) a_{k+2} + (k+1) a_{k+1} + a_{k-1} + a_k \right] z^k = 0$$

$$\Rightarrow a_{k+2} = \frac{-\left((k(k+1) + k+1) a_{k+1} + a_{k-1} + a_k \right)}{(k+2)(k+1)}$$

$$\boxed{a_{k+2} = \frac{-\left((k^2 + 2k + 1) a_{k+1} + a_{k-1} + a_k \right)}{(k+2)(k+1)}}$$

(*) Coeff. with negative subscripts are zero.

b) $k = 0, \quad a_2 = -\frac{a_1}{2} - \frac{a_0}{2}$

$$y = a_0 + a_1 (x-1) + \left(-\frac{a_1}{2} - \frac{a_0}{2} \right) (x-1)^2$$

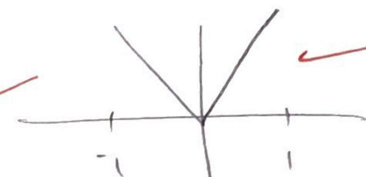
$$\boxed{y = a_0 \left(1 - \tfrac{1}{2} (x-1)^2 + \cdots \right) + a_1 \left((x-1) - \tfrac{1}{2} (x-1)^2 + \cdots \right)}$$

6. (10 points) Compute the Fourier series for the function f such that $f(x) = |x|$ on the interval $(-1, 1)$ and $f(x+2) = f(x)$.

$L = 1$

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Even



$$a_0 = 2\int_0^1 x\,dx = x^2\Big|_0^1 = 1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 2\int_0^1 x\cos\left(\frac{n\pi x}{1}\right)dx$$

$$a_n = 2\left(\cancel{\frac{x}{n\pi}\sin(n\pi x)}^{\to 0} + \frac{1}{n^2\pi^2}\cos(n\pi x)\right)\Big|_0^1$$

$$a_n = \frac{2}{n^2\pi^2}\cos(\pi x)^{\to(-1)^n} - \frac{2}{n^2\pi^2} \quad \begin{cases} 0, & n \text{ is even} \\ \dfrac{-4}{n^2\pi^2}, & n \text{ is odd} \end{cases}$$

D		I
x	+	$\cos(n\pi x)$
1	−	$\frac{1}{n\pi}\sin(n\pi x)$
0		$\frac{-1}{n^2\pi^2}\cos(n\pi x)$

$$\boxed{\mathcal{F}(f) = \frac{1}{2} + \sum_{1}^{\infty} \frac{-4}{(2n-1)^2\pi^2}\cos((2n-1)\pi x)}$$

20 mk

7. (10 points) Assume that y_1 and y_2 are solutions of the equation $y'' + py' + qy = 0$ where p and q are functions of t. Show that $c_1y_1 + c_2y_2$ is a solution of the equation for any values of the constants c_1, c_2.

Proof: Assume that y_1 and y_2 are solutions to $y'' + py' + qy = 0$.

That is, $y_1'' + py_1' + qy_1 = 0$ and $y_2'' + py_2' + qy_2 = 0$.

Show that $c_1y_1 + c_2y_2$ is also a solution.

Observe that $(c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2)$

$$= c_1y_1'' + c_2y_2'' + c_1py_1' + c_2y_2' + c_1qy_1 + c_2qy_2$$

by rearranging

$$= c_1\underbrace{(y_1'' + py_1' + qy)}_{0} + c_2\underbrace{(y_2'' + py_2' + qy_2)}_{0}$$

above is true regardless of the values of c_1 and c_2.

Thus, $c_1y_1 + c_2y_2$ is also a solution. $\square$

8. (10 points) Solve the IVP: $y'' + y' = t^2 + 2t$, $y(0) = 2$, $y'(0) = -1$.

y_h

$r^2 + r = 0 \quad r(r+1) = 0 \quad y_h = C_1 + C_2 e^{-t}$

$r = 0, -1$

$y_p = (At^2 + Bt + C)t = At^3 + Bt^2 + Ct$

$y' = 3At^2 + 2Bt + C$

$y'' = 6At + 2B$

$6At + 2B + 3At^2 + 2Bt + C = t^2 + 2t$

$2B + C = 0$

$6At + 2Bt = 2t \rightarrow 3A + B = 1$

$3At^2 = t^2 \rightarrow 3A = 1$

$A = 1/3, \ B = 0, \ C = 0$

$$\boxed{y = \frac{t^3}{3} + C_1 + C_2 e^{-t}}$$

find these by using I.C.

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9. (10 points) Compute the general solution of $y'' + y = \cot(t)$. $\rightarrow$ $y_h: r^2 + 1 = 0$

$r = \pm i$

$y_h = C_1 \underbrace{\cos(t)}_{y_1} + C_2 \underbrace{\sin(t)}_{y_2}$

$\boxed{W = 1}$

$u = -\int \frac{g y_2}{aW} dt = -\int \cot(t) \cdot \sin(t)\, dt$

$= -\int \cos(t)\, dt = \boxed{-\sin(t) + C_1}$

$v = \int \frac{g y_1}{aW} dt = \int \cot(t) \cdot \cos(t)\, dt = \int \frac{\cos^2(t)}{\sin(t)} dt$

$= \int \frac{1 - \sin^2(t)}{\sin(t)} dt = \int \csc(t)\, dt - \int \sin(t)\, dt$

$= \boxed{-\ln|\csc(t) + \cot(t)| + \cos(t) + C_2}$

So, $y = u y_1 + v y_2$

$= (-\sin(t) + C_1)\cos(t) + (-\ln|\csc(t) + \cot(t)| + \cos(t) + C_2)\sin(t)$

$$\Rightarrow \boxed{y = -\ln|\csc(t) + \cot(t)|\sin(t) + C_1\cos(t) + C_2\sin(t)}$$