

Math 39100 : Mar. 13. 2023 : LECTURE 12

§ 3.5 cont.

Gen. Soln: $y = y_h + y_p$

$$y = C_1 \cos(2t) + (C_2 \sin(2t)) + \sin(t)$$

$$y(0) = 2 \Rightarrow C_1$$

$$y'(0) = -1 \Rightarrow y' = -2C_2 \sin(2t) + 2C_1 \cos(2t) + \cos(t)$$

$$\Rightarrow -1 = -2(C_2)(0) + 2C_1 + 1$$

$$-1 = 2C_1 + 1$$

$$-1 = C_1$$

Solution to IVP:

$$y = 2 \cos(2t) - 1 \sin(2t) + \sin(t)$$

Summary of the Method Unde. Coeff:

Given $ay'' + by' + Cy = g(t)$ and y_h solution
to the corresponding homogeneous eq.

then, if

$$y(t)$$

Guess (y_p)

$$ae^{\alpha t}$$

$$Ae^{\alpha t}$$

$$a \sin(\beta t)$$

$$A \sin(\beta t) + B \cos(\beta t)$$

$$a \cos(\beta t)$$

$$A \sin(\beta t) + B \cos(\beta t)$$

$$at^2 + bt + c$$

$$At^2 + Bt + C$$

$$a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$$

$$A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0$$

$$e^{\alpha t} \cos(\beta t)$$

$$Ae^{\alpha t} \cos(\beta t) + Be^{\alpha t} \sin(\beta t)$$

If our guess for y_p is in Y_h , then we must multiply

y_p by t or t^2 or ... until it's gone.

In each of Problems 16 through 21:

y_p

- a. Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

- b. Use a computer algebra system to find a particular solution of the given equation.

16. $y'' + 3y' = 2t^4 + t^2 e^{-3t} + \sin(3t)$

$\underbrace{y_{p_1}}_{y_p}, \underbrace{y_{p_2}}_{y_{p_2}}, \underbrace{y_{p_3}}_{y_{p_3}}$

$$y_h : y'' + 3y' = 0$$

$$r^2 + 3r = 0 \Rightarrow r_1 = 0, r_2 = -3$$

$$y_h = C_1 + C_2 e^{-3t}$$

$$\begin{aligned}
 y_p &= t y_{p_1} + t y_{p_2} + y_{p_3} \\
 &= (At^4 + Bt^3 + Ct^2 + Dt + E)t \\
 &\quad + (Ft^2 + Gt + H)e^{-3t} \\
 &\quad + I \sin(3t) + J \cos(3t)
 \end{aligned}$$

3.6 Variation of parameters (General Method)

Start with $ay'' + by' + cy = g(t)$ (EQ1).

We know the solution to the homogeneous eq:

$$ay'' + by' + cy = 0 \quad (\text{EQ2})$$

y_1 and y_2 solve (eq.2)

$$y_h = C_1 y_1 + C_2 y_2$$

IDEA: Vary these parameters C_1 and C_2 .

lets use u and v instead C_1 and C_2 respectively.

Then our general soln : $y = u(t)y_1 + v(t)y_2$.

$$y = uy_1 + vy_2$$

$$\begin{aligned} y' &= u'y_1' + uy_1' + v'y_2' + vy_2' \\ &= \boxed{uy_1' + vy_2'} + \boxed{u'y_1 + v'y_2} \xrightarrow{0} \\ &\quad \text{Assumption 1: } u'y_1 + v'y_2 = 0 \end{aligned}$$

$$y'' = u'y_1'' + uy_1'' + v'y_2'' + vy_2''$$

Now, plug

y'' , y' and y into (eq 1) :

$$\begin{aligned} &a(uy_1' + uy_1'' + v'y_2' + vy_2'') + b(uy_1 + vy_2) \\ &+ c(uy_1 + vy_2) = g(\epsilon) \end{aligned}$$

$$\begin{aligned} &a uy_1'' + av y_2'' + au y_1' + av y_2' + bu y_1' + bv y_2' \\ &+ cu y_1 + cv y_2 = g(\epsilon) \end{aligned}$$

$$\begin{aligned} &au y_1' + av y_2' + au y_1'' + bu y_1' + cu y_1 + av y_2'' + bv y_2' \\ &\quad \cancel{+ cu y_1 + bv y_2} \xrightarrow{0} \\ &u(ay_1'' + by_1' + cy_1) + cv y_2 = g(\epsilon) \end{aligned}$$

Since y_1 and y_2 solves g_n .

$$a(u'y_1' + v'y_2') = g(t) \quad \text{Assumption #2.}$$

$$\begin{cases} u'y_1' + v'y_2' = 0 \\ a(u'y_1' + v'y_2') = g \end{cases}$$

Goal: Solve for u and v .

$$W = \begin{vmatrix} y_1 & y_2 \\ ay_1' & ay_2' \end{vmatrix} = a(y_1y_2' - y_2y_1')$$

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ g & ay_2' \end{vmatrix}}{aw} = \frac{-y_2g}{aw}$$

$$\Rightarrow u = -\int \frac{gy_2}{aw} dt \quad \leftarrow \text{Remember this}$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ ay_1' & g \end{vmatrix}}{aw} = \frac{gy_1}{aw}$$

$$\Rightarrow V = \int -\frac{gy_1}{aw} dt \quad \text{Remember this.}$$

So, general solution : $y = uy_1 + vy_2$

Find the general solution of

$$y'' + 4y = 8 \tan t \quad -\pi/2 < t < \pi/2.$$

Steps:

- 1) Find y_h . Then label y_1 and y_2 .
- 2) Compute W .
- 3) Find U and V
- 4) General soln: $y = uy_1 + vy_2$.

$$y_h: y'' + 4y = 0$$

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

$$y = C_1 \underbrace{\cos(2t)}_{y_1} + C_2 \underbrace{\sin(2t)}_{y_2}$$

$$W = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix}$$

$$= 2\cos^2(2t) + 2\sin^2(2t)$$

$$= 2(1)$$

$$= \boxed{2}$$

$$a=1, g(t) = 8 \tan t$$

$$u = - \int \frac{g y_2}{aw} dt$$

$$= - \int \frac{8 \cdot \tan t \cdot \sin(2t)}{2} dt$$

$$= -4 \int \tan(t) \sin(2t) dt$$

$$= -4 \cdot 2 \int \frac{\sin(t) \cdot \sin(t) \cos(t)}{\cos(t)} dt$$

$$= -8 \int \sin^2(t) dt$$

$$= -8 \int \frac{1}{2} (1 - \cos(2t)) dt$$

$$= -4 \left(t - \frac{\sin(2t)}{2} \right) + C_1$$

useful identities

$$(i) \sin(2x) = 2\sin x \cos x$$

$$(ii) \cos(2x) = \cos^2 x - \sin^2(x)$$

$$(iii) \sin^2(t) = \frac{1}{2}(1 - \cos(2t))$$

Now Compute v :

$$v = \int \frac{g y_1}{aw} dt = \int \frac{8 \cdot \tan(t) \cdot \cos(2t)}{2} dt$$

⋮

To be continued...