

Math 39100 : April. 17. 2023 : LECTURE 19

Exam 2: Monday, May 1 (Starting from 3.5, ...)

Exam 3: (optional) Friday, May 12

Time: 4 - 5:15 pm

Room: TBA

FINAL EXAM: May, 18 from 3:30 - 5:45 pm

Room: TBA

§ 5.4 cont.

Given $p(x)y'' + q(x)y' + r(x)y = 0$.

def: $x = x_0$ is a singular point if $p(x_0) = 0$.

if $\lim_{x \rightarrow x_0} (x - x_0) \cdot \frac{q(x)}{p(x)} = \text{finite}$ and

$\lim_{x \rightarrow x_0} (x - x_0)^2 \cdot \frac{r(x)}{p(x)} = \text{finite}$ then x_0 is a
Regular singular point.

Otherwise, irregular.

Determine the singular points of the Legendre equation

$$\underbrace{(1-x^2)y''}_{p(x)} - \underbrace{2xy'}_{q(x)} + \underbrace{\alpha(\alpha+1)y}_{r(x)} = 0$$

and determine whether they are regular or irregular.

$$p(x) = 0 \Rightarrow (1-x^2) = 0$$

$x = \pm 1$

singular points.

$$x_0 = 1 : \lim_{x \rightarrow 1} (x-1) \cdot \frac{(-2x)}{(1-x^2)} = \lim_{x \rightarrow 1} (x-1) \cdot \frac{-2x}{-1(1-x)(1+x)}$$

$$= \lim_{x \rightarrow 1} \frac{2x}{1+x} = \boxed{1} \text{ finite}$$

$$\lim_{x \rightarrow 1} (x-1)^2 \cdot \frac{\alpha(\alpha+1)}{-1(1+x)(1-x)} = \lim_{x \rightarrow 1} -1(x-1) \cdot \frac{\alpha(\alpha+1)}{1+x}$$

$$= \boxed{0} \text{ finite}$$

$\therefore x_0 = 1$ is a R.S.P.

$$x_0 = -1 : \lim_{x \rightarrow -1} (x+1) \frac{(-2x)}{(1-x^2)} = \lim_{x \rightarrow -1} \frac{-2x}{1-x} = \boxed{1}$$

$$\lim_{x \rightarrow -1} (x+1)^2 \cdot \frac{\alpha(\alpha+1)}{(1-x^2)} = \boxed{0} .$$

$\therefore x_0 = -1$ is also a R.S.P.

Why! Given $P(x)y'' + Q(x)y' + R(x)y = 0$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

Mult. both sides by x^2 :

$$x^2 y'' + x^2 \underbrace{\frac{Q}{P} y'}_{\beta} + x^2 \frac{R}{P} y = 0$$

$$x^2 y'' + x \left(x \frac{Q}{P} \right) y' + \left(x^2 \frac{R}{P} \right) y = 0$$

$\underbrace{\qquad}_{\beta} \qquad \underbrace{\qquad}_{\gamma}$

$x_0 = 0$

$$\lim_{x \rightarrow 0} x \cdot \frac{Q}{P} = \beta \quad \text{and} \quad \lim_{x \rightarrow 0} x^2 \frac{R}{P} = \gamma$$

§ 5.5 Series Solutions near Regular Singular Point

Given: $P(x)y'' + Q(x)y' + R(x)y = 0$

Solution near a regular singular point.

Assume solution form is: $y = x^r \sum_{n=0}^{\infty} a_n (x-x_0)^n$

Frobenius Method: assuming $x_0 = 0 \Rightarrow y = x^r \sum_{n=0}^{\infty} a_n x^n$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

then $y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$

and $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$

Goals : (i) Recurrence Formula of the coeff.

(ii) find value(s) for r : (come from

Solving the corresponding Euler equation)

(iii) Write the solution using the larger root

or smaller root.

$$\alpha x^2 y'' + \beta x y' + \gamma y = 0$$

Solve the differential equation

$$2x^2 y'' - xy' + (1+x)y = 0.$$

α β γ

$x_0 = 0$ is a singular point since $p(x) = 2x^2 = 0$ at $x=0$.

$$\lim_{x \rightarrow 0} x \cdot \frac{(-x)}{2x^2} = -\frac{1}{2} = \beta$$

$$\lim_{x \rightarrow 0} x^2 \cdot \frac{(1+x)}{2x^2} = \frac{1}{2} = \alpha$$

Finite $\Rightarrow \therefore x_0 = 0$ is R.S.P

$$\text{Corresponding Euler Eq: } x^2 y'' - \underbrace{\frac{1}{2}xy'}_{\text{S3}} + \underbrace{\frac{1}{2}y}_r = 0$$

$$2x^2 y'' - xy' + y = 0$$

$$2r(r-1) - r + 1 = 0$$

$$2r^2 - 3r + 1 = 0$$

$$(2r-1)(r-1) = 0$$

$$r = \frac{1}{2}, 1$$

$$\text{Sola form: } y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$2x^2 y'' - xy' + (1 + \overbrace{x}) \overbrace{y} = 0.$$

$$2x^2 y'' - xy' + y + xy = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r-1)(n+r) a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r} + x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2(n+r-1)(n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r}$$

$k=n$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$k=n+1 \Rightarrow n=k-1$$

$$\Rightarrow \sum_{k=0}^{\infty} 2(k+r-1)(k+r) a_k x^{k+r} - \sum_{k=0}^{\infty} (k+r) a_k x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r}$$

$$+ \sum_{k=1}^{\infty} a_{k-1} x^{k+r} = 0$$

$$\Rightarrow 2 \cdot (r-1)r a_0 x^r - r a_0 x^r + a_0 x^r$$

$$+ \sum_{k=1}^{\infty} [2(k+r-1)(k+r) a_k - (k+r) a_k + a_k + a_{k-1}] x^{k+r} = 0$$

$$x^r a_0 (2(r-1)r - r + 1)$$

to be continued ...