

Math 39100-Spring 2018 Final Exam solutions

1. Compute the general solution of each of the following:

a) $\frac{dy}{dx} = x + xy^2$

$$\frac{dy}{dx} = x + xy^2$$

$$\frac{dy}{dx} = x(1+y^2) \quad \text{SEPARABLE.}$$

$$\frac{1}{1+y^2} dy = x dx$$

$$\int \frac{1}{1+y^2} dy = \int x dx$$

$$\tan^{-1}(y) = \frac{1}{2}x^2 + C$$

b) $x \frac{dy}{dx} = x^3 + xy$

$$x \frac{dy}{dx} = x^3 + xy$$

$$\frac{dy}{dx} = x^2 + y \quad \text{LINEAR; INTEGRATING FACTOR}$$

$$\frac{dy}{dx} - y = x^2$$

$$\mu(x) = e^{\int p(x) dx} = e^{\int 1 dx} = e^{-x}$$

TABULAR METHOD

<u>der</u>	<u>INT</u>
x^2	e^{-x}
$2x$	$-e^{-x}$
2	e^{-x}
0	$-e^{-x}$

$$\int \frac{1}{dx} [e^{-x} y] = \int x^2 e^{-x} dx$$

$$e^{-x} y = -x^2 e^{-x} - 2x e^{-x} - 2 e^{-x} + C$$

$$y = -x^2 - 2x - 2 + Ce^x$$

c) $(x^2 + y^2) \frac{dy}{dx} = xy$

$$(x^2 + y^2) \frac{dy}{dx} = xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2 + y^2} \quad \text{1st order homogeneous}$$

$$\frac{dy}{dx} = \frac{\frac{xy}{x^2}}{\frac{x^2 + y^2}{x^2}}$$

$$\frac{dy}{dx} = \frac{\frac{y}{x}}{1 + (\frac{y}{x})^2}$$

Let $v = \frac{y}{x} \Rightarrow y = vx$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot x + v$$

$$v + x \frac{dv}{dx} = \frac{v}{1+v^2}$$

$$x \frac{dv}{dx} = \frac{v}{1+v^2} - v$$

$$x \frac{dv}{dx} = \frac{v - v - v^3}{1+v^2} = \frac{-v^3}{1+v^2}$$

$$\frac{1+v^2}{v^3} dv = -\frac{1}{x} dx$$

$$\int \frac{1+v^2}{v^3} dv = - \int \frac{1}{x} dx$$

$$\begin{aligned} \int \frac{1+y^2}{y^3} dy &= - \int \frac{1}{x} dx \\ \int \frac{1}{y^3} dy + \int \frac{1}{y} dy &= -\ln|x| + C \\ \int \left(\frac{1}{y^3} + \frac{1}{y} \right) dy &= -\ln|x| + C \\ -\frac{1}{2}y^{-2} + \ln|y| &= -\ln|x| + C \end{aligned}$$

plus in $y = \frac{u}{x}$

$$-\frac{1}{2}\left(\frac{u}{x}\right)^{-2} + \ln\left|\frac{u}{x}\right| = -\ln|x| + C$$

- 2) Solve the initial value problem: a) $e^{xy}(1+xy) + (x^2e^{xy} + 2y)y' = 0, \quad y(0) = 3$.

$$\underbrace{e^{xy}(1+xy)}_{M(x,y)} dx + \underbrace{(x^2e^{xy} + 2y)}_{N(x,y)} dy = 0, \quad y(0) = 3$$

$M_y = N_x$ THEN IT IS EXACT.

$$\begin{aligned} M_y &= x e^{xy}(1+xy) + e^{xy}(x) = x e^{xy} + x^2 y e^{xy} + x e^{xy} \\ N_x &= 2x e^{xy} + x^2 e^{xy} \cdot y \end{aligned}$$

$\Rightarrow M_y = N_x \therefore \text{EXACT!}$

$$\begin{aligned} f(x,y) &= \int e^{xy} + x y e^{xy} dx \\ &= \frac{1}{y} e^{xy} + y \cdot x \cdot \frac{1}{y} e^{xy} - y \cdot \frac{1}{y^2} e^{xy} + g(y) \\ &= \frac{1}{y} e^{xy} + x e^{xy} - \frac{1}{y} e^{xy} + g(y) \end{aligned}$$

$$\begin{aligned} f(x,y) &= x e^{xy} + g(y) \quad \leftarrow \\ g'(y) &= x^2 e^{xy} + g'(y) = N = x^2 e^{xy} + 2y \\ g'(y) &= 2y \\ \Rightarrow g(y) &= y^2 \end{aligned}$$

$$\begin{aligned} \text{SOLN: } x e^{xy} + y^2 &= C \\ y(0) = 3 &\Rightarrow 0 e^{0 \cdot 3} + 3^2 = C \\ &\Rightarrow C = 9 \end{aligned}$$

$$\boxed{\text{SOLN: } x e^{xy} + y^2 = 9}$$

- (1d) Compute the general solution of $y'' + 6y' + 9y = t^2 e^{-3t}$

Find $y_h: r^2 + 6r + 9 = 0$

$$\Rightarrow (r+3)^2 = 0$$

$r = -3$ REPEATED.

$$\text{So, } y_h = C_1 e^{-3t} + C_2 t e^{-3t}$$

$$y_p = (A t^2 + B t + C) e^{-3t} \cdot t^2$$

$$\Rightarrow y_p = A t^4 e^{-3t} + B t^3 e^{-3t} + C t^2 e^{-3t}$$

$$y_p' = 4At^3 e^{-3t} - 3At^4 e^{-3t} + 3Bt^2 e^{-3t} - 3Bt^3 e^{-3t} + 2Ct e^{-3t} - 3Ct^2 e^{-3t}$$

$$= (4A - 3B)t^3 e^{-3t} - 3At^4 e^{-3t} + (3B - 3C)t^2 e^{-3t} + 2Ct e^{-3t}$$

$$y_p'' = 3(4A - 3B)t^2 e^{-3t} - 3(4A - 3B)t^3 e^{-3t} \dots$$

⋮

SORRY ALL "1". SEEKS LIKE I MADE A TYPING MISTAKE WHEN I TYPED THIS

PROBLEM. [FYI: THIS IS A VERY LONG PROBLEM TO DETERMINE THESE UNKNOWN CONSTANTS A, B , & C .]

ONLY FOR THIS PROBLEM WITHOUT SOLVING FOR A, B, C , OUR GENERAL SOLUTION IS:

$$y = C_1 e^{-3t} + C_2 t e^{-3t} + At^4 e^{-3t} + Bt^3 e^{-3t} + Ct^2 e^{-3t}$$

(2b) Solve the initial value problem: $y'' + 4y' + 4y = 4 - 12e^{-2t}$, $y(0) = 0$, $y'(0) = 0$.

$$y_h: r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \Rightarrow r = -2 \quad \text{REPEATED}$$

$$\text{so, } y_h = C_1 e^{-2t} + C_2 t e^{-2t}$$

$$y_p = A + B t^{-2} e^{-2t}$$

$$= A + B t^2 e^{-2t}$$

$$\text{Then, } y_p' = 2Bt e^{-2t} - 2Bt^2 e^{-2t}$$

$$y_p'' = 2B e^{-2t} - 4Bt e^{-2t} - 4Bt e^{-2t} + 4Bt^2 e^{-2t}$$

$$= 2B e^{-2t} - 8Bt e^{-2t} + 4Bt^2 e^{-2t}$$

NOW PLUG IN y_p'' , y_p' AND y_p INTO GIVEN EQUATION YIELDS :

$$2B e^{-2t} - 8Bt e^{-2t} + 4Bt^2 e^{-2t} + 8Bt e^{-2t} - 8Bt^2 e^{-2t} + 4A + 4Bt^2 e^{-2t} = 4 - 12e^{-2t}$$

$$\Rightarrow 4A = 4 \Rightarrow A = 1$$

$$2B = -12 \Rightarrow B = -6$$

THEREFORE THE GENERAL SOLUTION IS :

$$y = C_1 e^{-2t} + C_2 t e^{-2t} + 1 - 6t^2 e^{-2t}$$

NOW APPLY INITIAL CONDITIONS : $y(0) = 0$, $y'(0) = 0$

$$y(0) = 0 \Rightarrow 0 = C_1 + 1 \Rightarrow C_1 = -1$$

$$y' = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 t e^{-2t} - 12t e^{-2t} + 12t^2 e^{-2t}$$

$$\text{so, } y'(0) = 0 \Rightarrow 0 = 2 + C_2 \Rightarrow C_2 = -2$$

Hence, $y = -e^{-2t} - 2t e^{-2t} + 1 - 6t^2 e^{-2t}$

(3) A tank with a capacity of 200 gallons initially contains 50 gallons of a solution containing 3 pounds of salt. Fresh water enters at a rate of 5 gallons per minute and a well-stirred mixture is pumped out at the rate of 2 gallon per minute. Find an expression for the amount of salt $Q(t)$ (in pounds) in the tank t minutes after the process begins, up to the moment when the tank is full. Use it to compute the amount of salt in the tank at the moment when the tank is filled up.

GIVEN:

$$V_i = 50$$

$$Q(0) = 3$$

$$r_{in} = 5$$

$$C_{in} = 0$$

$$r_{out} = 2$$

$$\begin{aligned}\frac{dQ}{dt} &= r_{in} C_{in} - r_{out} C_{out} \\ &= (5)(0) - 2 \cdot \frac{Q(t)}{50 + 3t}\end{aligned}$$

When is the tank

FULL?

$$200 = 50 + 3t$$

$$\Rightarrow t = 50$$

$$\frac{dQ}{dt} = -\frac{2Q(t)}{50 + 3t}$$

$$\int \frac{1}{Q(t)} dQ = \int -\frac{2}{50 + 3t} dt$$

$$\ln(Q(t)) = -\frac{2}{3} \ln(50 + 3t) + C$$

$$\Rightarrow Q(t) = C(50 + 3t)^{-\frac{2}{3}}, \quad 0 \leq t \leq 50$$

$$Q(0) = 3 \Rightarrow 3 = \frac{C}{50^{\frac{2}{3}}} \Rightarrow C = 3 \cdot 50^{\frac{2}{3}}$$

THUS,

$$Q(t) = \frac{3 \cdot 50^{\frac{2}{3}}}{(50 + 3t)^{\frac{2}{3}}}, \quad 0 \leq t \leq 50$$

THE TANK IS FULL WHEN $t = 50$,

$$Q(50) = \frac{3 \cdot 50^{\frac{2}{3}}}{(200)^{\frac{2}{3}}}$$

(4) For the differential equation:

$$(2-x)y'' + (3-x^2)y' + 5x^2y = 0$$

Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$ and use it to compute the first four nonzero terms of the solution with $y(0) = 6$, $y'(0) = 0$.

MISSING = SIGN

ASSUME SOLN FORM: $y = \sum a_n x^n$, THEN $y' = \sum n a_n x^{n-1}$, $y'' = \sum n(n-1) a_n x^{n-2}$. THEN

$$2 \sum n(n-1) a_n x^{n-2} - \sum n(n-1) a_n x^{n-1} + 3 \sum n a_n x^{n-1} - \sum n a_n x^{n+1} + 5 \sum a_n x^{n+2} = 0$$

$$2 \sum_{k=n-2}^n n(n-1) a_n x^{n-2} - \sum_{k=n-1}^n n(n-1) a_n x^{n-1} + 3 \sum_{k=n-1}^n n a_n x^{n-1} - \sum_{k=n+1}^n n a_n x^{n+1} + 5 \sum_{k=n+2}^n a_n x^{n+2} = 0$$

$$\Rightarrow \sum_{k=2}^{k+2} (k+2)(k+1) a_{k+2} x^k - \sum_{k=1}^{k+1} (k+1) k a_{k+1} x^k + \sum_{k=1}^{k+1} 3(k+1) a_{k+1} x^k - \sum_{k=1}^{k-1} (k-1) a_{k-1} x^k + 5 \sum_{k=2}^{k-2} a_{k-2} x^k = 0$$

$$\Rightarrow \sum [2(k+2)(k+1)a_{k+2} - (k+1)k a_{k+1} + 3(k+1) a_{k+1} - (k-1) a_{k-1} + 5 a_{k-2}] x^k = 0$$

$$\Rightarrow a_{k+2} = \frac{k(k+1)a_{k+1} - 3(k+1)a_{k+1} + (k-1)a_{k-1} - 5a_{k-2}}{2(k+2)(k+1)}$$

$$\Rightarrow a_{k+2} = \frac{(k^2 + k - 3k - 3)a_{k+1} + (k-1)a_{k-1} - 5a_{k-2}}{2(k+2)(k+1)}$$

$$a_{k+2} = \frac{(k-3)(k+1)a_{k+1} + (k-1)a_{k-1} - 5a_{k-2}}{2(k+2)(k+1)} \quad \text{Recursion Formula}$$

Given:

$$\left. \begin{array}{l} y(0) = -6 = a_0 \\ y'(0) = 0 = a_1 \end{array} \right| \quad \begin{array}{l} k=0, a_2 = \frac{(-3)(1)a_1 + (-1)a_{-1} - 5a_{-2}}{2(2)(1)} = 0 \\ k=1, a_3 = \frac{(-2)(2)a_2 + 0 - 5a_{-1}}{2(3)(2)} = 0 \end{array}$$

$$k=2, a_4 = \frac{(-1)(3)a_3^{\circ} + 1a_1^{\circ} - 5a_0}{2(4)(3)} = \frac{-5(-6)}{2 \cdot 4 \cdot 3} = \frac{5}{4}$$

$$k=3, a_5 = \frac{0 + 2a_2^{\circ} - 5a_1^{\circ}}{2(5)(4)} = 0$$

$$k=4, a_6 = \frac{(1)(5)a_5^{\circ} + 3a_3^{\circ} - 5a_2^{\circ}}{2(6)(5)} = 0$$

$$k=5, a_7 = \frac{(2)(6)a_6^{\circ} + 4a_4^{\circ} - 5a_3^{\circ}}{2(7)(6)} = \frac{4a_4}{2 \cdot 7 \cdot 6} = \frac{4 \cdot \frac{5}{4}}{2 \cdot 7 \cdot 6}$$

$$= \frac{5}{84} \quad \text{AWESOME !!}$$

$$k=6, a_8 = \frac{(3)(7)a_7^{\circ} + 5a_5^{\circ} - 5a_4^{\circ}}{2(8)(7)}$$

$$= \frac{3 \cdot 7 \cdot a_7}{2 \cdot 8 \cdot 7} - \frac{5 a_4}{2 \cdot 8 \cdot 7} = \frac{3 a_7}{16} - \frac{5 \cdot a_4}{2 \cdot 8 \cdot 7}$$

$$= \frac{3 \cdot 5}{16 \cdot 84} - \frac{5 \cdot 5}{2 \cdot 8 \cdot 7 \cdot 4}$$

Soln: $y = a_0 + a_1 x^0 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + \dots$

Therefore, $y = -6 + \frac{5}{4} x^4 + \frac{5}{84} x^7 + \left(\frac{15}{16 \cdot 84} - \frac{25}{64 \cdot 7} \right) x^8 + \dots$

(7) (a) State the definition of the Laplace Transform and use it to compute the Laplace Transform of the function f with $f(t) = 1$.

(b) Compute the Laplace Transform of the solution to the initial value problem:

$$y'' - 5y' + 6y = 10$$

with $y(0) = 2$, $y'(0) = -3$.

(a) DEF: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t) dt$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 dt = \lim_{T \rightarrow \infty} \int_0^T e^{-st} dt$$

$$= \lim_{T \rightarrow \infty} \left[-\frac{1}{s} e^{-st} \right] \Big|_0^T$$

$$= \lim_{T \rightarrow \infty} \left[-\frac{1}{s} e^{-sT} + \frac{1}{s} \right]$$

$$= \boxed{\frac{1}{s}}$$

(b) $y'' - 5y' + 6y = 10$, $y(0) = 2$, $y'(0) = -3$

$$\mathcal{L}\{y''\} - 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{10\}$$

$$\Rightarrow s^2 \mathcal{L}\{y\} - s y(0) - y'(0)s^{-1} - 5s \mathcal{L}\{y\} + 5 y(0)s^{-1} + 6 \mathcal{L}\{y\} = \frac{10}{s}$$

$$\Rightarrow \mathcal{L}\{y\}(s^2 - 5s + 6) - 2s + 3 + 10 = \frac{10}{s}$$

$$\Rightarrow \mathcal{L}\{y\} = \frac{\frac{10}{s} + 2s - 13}{s^2 - 5s + 6} = \frac{10 + 2s^2 - 13s}{s(s^2 - 5s + 6)}$$

$$\Rightarrow \boxed{\mathcal{L}\{y\} = \frac{2s^2 - 13s + 10}{s(s-3)(s-2)}} \quad \text{OR} \quad (\text{SET-UP THE PARTIAL FRACTION DECOMPOSITION!})$$

$$\frac{2s^2 - 13s + 10}{s(s-3)(s-2)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-2}$$

$$\Rightarrow 2s^2 - 13s + 10 = A(s-3)(s-2) + B \cdot s(s-2) + C \cdot s(s-3)$$

$$\text{LET } s=0, \Rightarrow 10 = A \cdot (-3)(-2) \Rightarrow \frac{10}{6} = A \Rightarrow A = \frac{5}{3}$$

$$s=3, \Rightarrow 18 - 39 + 10 = B \cdot 3 \cdot (1)$$

$$-11 = 3B \Rightarrow B = -\frac{11}{3}$$

$$s=2 \Rightarrow 8 - 26 + 10 = C \cdot (2)(-1)$$

$$-8 = -2C \Rightarrow C = 4$$

SO, THEN

$$\boxed{\mathcal{L}\{y\} = \frac{5/3}{s} + \frac{-11/3}{s-3} + \frac{4}{s-2}}$$