

1. (10 points) The particular solution of the differential equation $y'' + 4y' + 4y = t^{-2}e^{-2t}$ is

a) $y_p = e^{-t} + 2e^{3t}$

b) $y_p = 2e^{-2t}$

c) $y_p = -e^{-2t} \ln(t)$

d) $y_p = e^{-2t} + 4 - e^{-2t} \ln(t)$

e) none of the above.

$$Y = uY_1 + vY_2$$

$$Y_h$$

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \quad r_1 = r_2 = -2$$

$$Y_h \Rightarrow c_1 e^{-2t} + c_2 t e^{-2t} = 0$$

$$W = \begin{vmatrix} e^{-2t} & t e^{-2t} \\ -2e^{-2t} & -2t e^{-2t} + e^{-2t} \end{vmatrix}$$

$$W = \frac{-2t e^{-4t} + e^{-4t} + 2t e^{-4t}}{e^{-4t}}$$

$$W = e^{-4t}$$

$$u = - \int \frac{g Y_2}{aw} = - \int \frac{t^{-2} e^{-2t} (t e^{-2t})}{e^{-4t}} dt$$

$$u = - \int \frac{1}{t} dt = -\ln|t| + c_1$$

$$v = \int \frac{g Y_1}{aw} = \int \frac{t^{-2} \cdot e^{-2t} \cdot e^{-2t}}{e^{-4t}} dt$$

$$v = -\frac{1}{t} + c_2$$

$$Y_p = \boxed{-\ln(t) \cdot e^{-2t} - e^{-2t}}$$

$$\boxed{Y_p = -e^{-2t} (\ln(t) + 1)}$$

2. (10 points) The general solution of the differential equation $x^2y'' + 3xy' - 35y = 0$ with $x > 0$ is

a) $y(x) = c_1 x^5 + c_2 x^{-7}$

let $y = x^r$

b) $y(x) = c_1 x^7 + c_2 x^{-5}$

$$r(r-1) + 3r - 35 = 0$$

c) $y(x) = c_1 x^5 + c_2 x^5 \ln(x)$

$$r^2 + 2r - 35 = 0$$

d) $y(x) = c_1 x^3 + c_2 x^{-5}$

$$(r+7)(r-5) = 0$$

e) none of the above.

$$Y = c_1 x^{-7} + c_2 x^5$$

3. (10 points) For the given differential equation:

$$(1 - x^2)y'' + 2xy' + y = 0$$

We seek a power series solution of the form $y = \sum_{n=0} a_n x^n$. The recursion formula is given by

a) $a_{k+2} = \frac{(k^2 + 3k + 1)a_k}{(k+2)(k+1)}$

$$y'' - x^2 y''' + 2xy' + y = 0$$

b) $\checkmark a_{k+2} = \frac{(k^2 - 3k - 1)a_k}{(k+2)(k+1)}$

$$\sum a_n x^{n-2} (n)(n-1) - \sum a_n x^n (n)(n-1) +$$

~~c)~~ $a_{k+1} = \frac{(k^2 + 3k - 1)a_k}{k(k+2)}$

$$\sum 2a_n x^n (n) + \sum a_n x^n = 0$$

✓

~~d)~~ $a_k = \frac{k(k+2)a_{k+2}}{k^2 - 3k - 1}$

$$\sum a_{k+2} x^k (k+2)(k+1) - \sum a_k x^k (k)(k-1) +$$

e) none of the above.

$$\sum 2a_n x^k (k) + \sum a_n x^k = 0$$

$$a_{k+2} = \frac{-a_k - 2a_{k+1} + a_k(k)(k-1)}{(k+2)(k+1)}$$

$$\boxed{a_{k+2} = \frac{-a_k(-1 - 2k + k^2 - k)}{(k+2)(k+1)}}$$

$$\boxed{\frac{a_k(k^2 - 3k - 1)}{(k+2)(k+1)}}$$

4. (5 points) [Part I] Which of the following represents the general solution of the differential equation:

$$2y^{(4)} - 5y^{(3)} - 12y'' = 0$$

(Assume c_1, c_2, c_3 , and c_4 are arbitrary constants.)

a) $y = c_1 + c_2 e^{4x} + c_3 e^{-3x/2}$

$$2r^4 - 5r^3 - 12r^2$$

✓ b) $y = c_1 + c_2 x + c_3 e^{4x} + c_4 e^{-3x/2}$

$$r^2(2r^2 - 5r - 12) = 0$$

c) $y = c_1 + c_2 x + c_3 \sin(3x/2) + c_4 \cos(3x/2)$

$$r = \frac{5 \pm \sqrt{25 - 4(2)(-12)}}{4}$$

d) $y = c_1 + c_2 x + c_3 e^{-4x} + c_4 e^{-2x/3}$

e) none of the above.

$$r_1 = r_2 = 0$$

$$r_3 = 4 \quad r_4 = -\frac{3}{2}$$

$$r = \frac{5}{4} \pm \frac{11}{4}$$

$$r = 4, -\frac{3}{2}$$

$$y = c_1 + x c_2 + c_3 e^{4x} + c_4 e^{-\frac{3}{2}x}$$

[PART II] (5 points) Determine the test function $Y(t)$ with the fewest terms to be used to obtain a particular solution of the following equation via the method of undetermined coefficients. Do not attempt to determine the values of the coefficients.

$$2r^4 - 5r^3 - 12r^2 = 0 \quad 2y^{(4)} - 5y^{(3)} - 12y'' = 7x^2 - 12xe^{4x} + \sqrt{3} \cos(3x/2) + 8$$

$$Y_h = c_1 + x c_2 + c_3 e^{4x} + c_4 e^{-\frac{3}{2}x}$$

$$Y_{P_1} \rightarrow 7x^2 + 8$$

$$Y_{P_2} = (Dx + E)e^{4x} \cdot x$$

$$Y_{P_2} \rightarrow -12x e^{4x}$$

$$Y_{P_3} = F \cos(\frac{3x}{2}) + G \sin(\frac{3x}{2})$$

$$Y_{P_3} \rightarrow \sqrt{3} \cos(\frac{3x}{2})$$

$$Y(t) = A x^4 + B x^3 + C x^2 + D x^2 e^{4x} + E x e^{4x} + F \cos(\frac{3x}{2}) + G \sin(\frac{3x}{2})$$

5. (10 points) Solve the initial value problem by using the Laplace Transform:

$$y'' - 4y = 3, \quad y(0) = 0, \quad y'(0) = 4$$

[No credit will be given for using any other method.]

$$\{y'' - 4y\} = \{3\} \quad \checkmark$$

$$s^2\{y\} - s\{y(0)\} - y'(0) - 4\{y\} = \frac{3}{s} \quad \checkmark$$

$$\{y\}(s^2 - 4) = \frac{3}{s} + 4 = \frac{3}{s} + \frac{4s}{s} = \frac{4s+3}{s}$$

$$\{y\} = \frac{4s+3}{s(s-2)(s+2)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+2}$$

$$A(s-2)(s+2) + B(s(s+2)) + C(s(s-2)) = 4s+3$$

$$s=2 \rightarrow B \cdot 8 = 11 \quad B = \frac{11}{8} \quad \{y\} = \frac{-3/4}{s} + \frac{11}{8} \cdot \frac{1}{s-2} + \frac{-s}{8} \frac{1}{s+2}$$

$$s=-2 \rightarrow C \cdot 8 = -5 \quad C = \frac{-5}{8}$$

$$s=0 \rightarrow A(-4) = 3 \quad A = \frac{-3}{4}$$

$$y = \frac{-3}{4} + \frac{11}{8} e^{2t} - \frac{5}{8} e^{-2t}$$

Good!



6. (10 points) A hanging spring is stretched 2 inches by a mass weighing 8 pounds. It is pulled down an additional foot and is released but continuously subjected to an external force of $4 \sin(7t)$. Set up the initial value problem (differential equation and initial conditions) which describes the motion, neglecting friction. (You do need not solve the equation.)

$$L = 2 \text{ in} = \frac{1}{6} \text{ ft} \quad u(0) = 1 \text{ ft}$$

$$W = 8 \text{ lb} \quad \checkmark \quad F(t) = 4 \sin 7t$$

$$m = \frac{W}{g} = \frac{8}{32} = \frac{1}{4} \quad k = \frac{8}{\frac{1}{6}} = 48$$

$$\boxed{\frac{1}{4}u'' + 48u = 4 \sin(7t); \quad u(0) = 1 \quad u'(0) = 0} \quad \checkmark$$

7. 10 points) Find the fourth term of the power series solution of the given IVP centered at $x_0 = 0$:

$$y'' + (\sin(x))y' + e^{2x}y = 0; \quad y(0) = 1, \quad y'(0) = 2$$

$$y = a_0 + a_1 x + a_2 x^2 + \underbrace{a_3 x^3}_{\text{we want this term}} + \dots$$

$$a_2: \quad y'' = -\sin(x)y' - e^{2x}y \Big|_{x_0=0} \Rightarrow y'' = -\sin(0)y'(0) - e^{2(0)}y(0) \\ = -1$$

$$a_3: \quad y''' = -\cos(x)y' - \sin(x)y'' - 2e^{2x}y - e^{2x}y' \Big|_{x_0=0} \\ = -\cos(0)y'(0) - \sin(0)y''(0) - 2e^{2(0)}y(0) - e^{2(0)}y'(0) \\ = -1 \cdot 2 - 0 - 2 \cdot 1 - 1 \cdot 2 = -6$$

$$\text{So, } a_3 = \frac{y'''(x_0)}{3!} = -\frac{6}{3!} = -1$$

Therefore, the Fourth term is

$$-x^3$$

8. (10 points) State the definition of the Laplace transform of a function $f(t)$ and use it to find the Laplace transform of the function $f(t) = \begin{cases} 8, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \leftarrow \text{Definition}$$

$$\int_0^1 e^{-st} \cdot 8 dt + \int_1^\infty e^{-st} \cdot 0 dt$$

$$8 \frac{e^{-st}}{-s} \Big|_0^1$$

$$\left[\frac{8e^{-s}}{-s} + \frac{8}{s} \right]_-$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{8e^{-s}}{-s} + \frac{8}{s}}$$

$$\lim_{b \rightarrow \infty} C_1(b) = b - \sqrt{b^2 - 8s}$$

5.

9. Consider the differential equation $x^2y'' + xy = 0$.

i) (3 points) Show that $x_0 = 0$ is a regular singular point.

ii) (7 points) Compute the recursion formula for the series solution corresponding to the larger root of the indicial equation. With $a_0 \neq 0$, write down the first three nonzero terms of the series solution.

i) Assume $P(x)y'' + Q(x)y' + R(x)y = 0$

$$P(x_0) = x_0^2 = 0 \quad \checkmark \text{ singular point}$$

$$\lim_{x \rightarrow 0} \frac{0}{(x^2)} \cdot x = 0 \quad \checkmark \quad \left. \begin{array}{l} \beta \\ \text{regular} \end{array} \right\} \quad \checkmark$$

$$\lim_{x \rightarrow 0} \frac{x}{x^2} \cdot x^2 = 0 \quad \checkmark$$

ii) $\frac{\text{Indicial Eq}}{r(r-1)} = 0$ Assume $y = \sum a_n x^{n+r} = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \dots$

$$r^2 - r = 0$$

$$r(r-1) = 0$$

$$r = 1, 0$$

$$\sum a_n x^{n+r} (n+r)(n+r-1) + \sum a_n x^{n+r+1} = 0$$

$$k = n$$

$$k = n+1$$

\checkmark

$$\sum a_n x^{k+r} (k+r)(k+r-1) + \sum a_{n-1} x^{k+r} = 0$$

$$a_k = \frac{-a_{k-1}}{(k+r)(k+r-1)}$$

$$r=1 \rightarrow a_n = \frac{-a_{n-1}}{(n+1)n}$$

↳ larger root

$$a_1 = \frac{-a_0}{2} \quad a_2 = \frac{-a_1}{6} = \frac{a_0}{12}$$

$$a_3 = \frac{-a_2}{12} = \frac{-a_0}{144}$$

$$\boxed{y = a_0 x - \frac{a_0 x^2}{2} + \frac{a_0 x^3}{12} - \dots}$$

Good job!

\checkmark