MATHEMATICS 391 **FINAL EXAM** **Spring, 2018**

NAME ___________________________

SECTION______________

INSTRUCTOR___________________________

Do all your work in this answer booklet. If you need extra space, use the facing pages.

Part I Answer all questions in this part.

(1) Compute the general solution of each of the following (6 points each):

(a) $\dfrac{dy}{dx} = x + xy^2.$

(b) $x \dfrac{dy}{dx} = x^3 + xy.$

(c) $(x^2 + y^2)\ \dfrac{dy}{dx} = xy.$

(d) $y'' + 6y' + 9y = t^{-2}e^{-3t}.$

(2) Solve the following initial value problems (6 points each):

(a) $e^{xy}(1 + xy) + (x^2 e^{xy} + 2y)y' = 0 \qquad y(0) = 3$

(b) $y'' + 4y' + 4y = 4 - 12e^{-2t}, \quad y(0) = 0,\ y'(0) = 0.$

(3)(8 points) A tank with a capacity of 200 gallons initially contains 50 gallons of a solution containing 3 pounds of salt. Fresh water enters at a rate of 5 gallons per minute and a well-stirred mixture is pumped out at the rate of 2 gallon per minute. Find an expression for the the amount of salt $Q(t)$ (in pounds) in the tank t minutes after the process begins, up to the moment when the tank is full. Use it to compute the amount of salt in the tank at the moment when the tank is filled up.

(4) (8 points)For the differential equation:

$$(2-x)y'' \; + \; (3-x^2)y' \; + \; 5x^2y \; = \; 0$$

Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$ and use it to compute the first four nonzero terms of the solution with $y(0) = -6, y'(0) = 0$.

(5) (a) (5 points) Compute the sine series for the function f such that $f(x) = 1 - x^2$ on the interval $[0, 1]$.

(b) (3 points) Compute the solution $u(x,t)$ for the partial differential equation with x in the interval $[0, 1]$ and $t > 0$:

$$\begin{aligned} u_t &= 49u_{xx} \quad \text{with} \\ u(0,t) = u(1,t) &= 0 \quad \text{for } t > 0 \qquad \text{(boundary conditions)} \\ u(x,0) &= 1 - x^2 \quad \text{for } 0 < x < \pi \qquad \text{(initial conditions)} \end{aligned}$$

PartII Answer all sections of four (4) questions out of the six (6) questions in this part (10 points each).

(6) For the equation $4x^2y'' \;-\; y \;=\; 0 \quad (t > 0)$, $y_1(x) = x^{1/2}$ is a solution.

(a) Use the method of Reduction of Order to obtain a second, independent solution.

(b) Solve the equation directly, using that it is an Euler Equation.

(c) Compute the Wronskian of the pair of solutions.

(7) (a) State the definition of the Laplace transform and use it to compute the Laplace transform of the function f with $f(t) = 1$.

Compute the Laplace transform of the solution to the initial value problem $y'' - 5y' + 6y = 10$ with $y(0) = 2, y'(0) = -3$.

(b) A car-buyer obtains an \$8000 loan with an annual interest rate of 5%. The borrower makes payments continuously at a rate of §1200 per year. Set up the initial value problem whose solution is the amount owed t years after the loan was made and up to when the loan is paid off. (You need not solve the equation).

(8) (a) Compute the general solution of the differential equation

$$y^{(7)} \;+\; 3y^{(5)} \;-\; 4y^{(3)} \;=\; 0.$$

(b) Determine the test function $Y(t)$ with the fewest terms to be used to obtain a particular solution of the following equation via the *method of undetermined coefficients*. Do not attempt to determine the coefficients.

$$y^{(7)} \;+\; 3y^{(5)} \;-\; 4y^{(3)} \;=\; 7t^3 \;+\; 5e^t \;-\; 2\cos(2t) + 8te^t\cos(2t) \;+\; 5.$$

(9) For the differential equation $2(x^2-x^4)y''+3xy'-(1+2x^2)y=0$ show that the point $x=0$ is a regular singular point (either by using the limit definition or by computing the associated Euler equation). Compute the recursion formula for the series solution corresponding to the larger root of the indicial equation. With $a_0=1$, write down the first three nonzero terms of the series.

(10) (a) Compute the Wronskian of the functions u_1 and u_2 given by $u_1(t) = t\cos(t), u_2(t) = t\sin(t)$. Use it to explain why these two cannot be a pair of solutions on the entire real line of an equation equation $y'' + py' + qy = 0$ where p and q are functions of t (Hint: Think about Abel's Theorem).

(b) Assume that y_1 and y_2 are solutions of the equation $y'' + py' + qy = 0$ where p and q are functions of t. Show that $C_1y_1 + C_2y_2$ is a solution of the equation for any values of the constants C_1, C_2.

(11) (a) Use the method of separation of variables to replace the equation $u_{xx} + 5u_{xt} - u_t = 0$ by a pair of ordinary differential equations.

(b) A hanging spring is stretched 4 inches by a mass weighting of 8 pounds. It is pulled down an additional foot and is released but continuously subjected to an external force of $6\cos(4t)$.

Set up the initial value problem (differential equation and initial conditions) which describes the motion, neglecting friction. You need not solve the equation. (Recall that g, the acceleration due to gravity is 32 feet/second^2.)