

Math 39100 : May. 3 . 2023 : LECTURE 23

Exam 3: (optional) Friday, May 12

Time: 4 - 5:15 pm

Room: Nac 1/511

FINAL EXAM: May, 18 from 3:30 - 5:45 pm

Room: Nac 0/201

Ch 10 : Partial Diff. Eq. and Fourier Series

10.1 Boundary Value Problems (BVP)

$$y'' + p(x)y' + q(x)y = g(x) \quad \text{with}$$

$$\text{Boundary Conditions} : \quad y(\alpha) = y_0 \quad \text{and} \quad y'(\beta) = y'_0$$

Recall:

$$\text{IVP: } y'' + p(x)y' + q(x)y = g(x)$$

$$\text{I.C: } y(\alpha) = y_0 \quad \text{and} \quad y'(\alpha) = y'_0$$

e.g. Solve BVP : $y'' + 2y = 0$, $y(0) = 1$, $y(\pi) = 0$

$$r^2 + 2 = 0$$

$$r = \pm \sqrt{2}i$$

$$y = C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x)$$

B.C (1) : $y(0) = 1 \Rightarrow \boxed{1 = C_1}$

$$y = \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x)$$

B.C (2) : $y(\pi) = 0 \Rightarrow 0 = \cos(\sqrt{2}\pi) + C_2 \sin(\sqrt{2}\pi)$

$$-\frac{\cos(\sqrt{2}\pi)}{\sin(\sqrt{2}\pi)} = \underbrace{-\cot(\sqrt{2}\pi) = C_2}_{C_2 = -\cot(\sqrt{2}\pi)}$$

Soln: $\boxed{y = \cos(\sqrt{2}x) - \cot(\sqrt{2}\pi) \sin(\sqrt{2}x)}$

e.g. Solve BVP: $y'' + 2y = 0$ $y(0) = 0$ and

$$y(\pi) = 0.$$

$$y = C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x)$$

B.C : $y(0) = 0 \Rightarrow \boxed{C_1 = 0}$

B.C : $y(\pi) = 0 \Rightarrow 0 = C_2 \sin(\sqrt{2}\pi)$

$$\boxed{C_2 = 0}$$

Soln: $y = 0$ Trivial Solution

e.g. Solve BVP: $y'' + y = 0$ $y(0) = 1$ and $y(L) = 0$.

$$y = C_1 \cos(x) + C_2 \sin(x)$$

$$y(0) = 1 \Rightarrow C_1 = 1$$

$$y = \cos(x) + C_2 \sin(x)$$

$$y(L) = 0 \Rightarrow 0 = \cos(L) + C_2 \sin(L)$$

$$C_2 = -\frac{\cos(L)}{\sin(L)} = -\cot(L)$$

$$y(x) = \cos(x) - \cot(L) \sin(x)$$

Goal: Solve $y'' + \lambda y = 0$, $y(\alpha) = 0$
 $y(\beta) = 0$

Recall: (from Linear Algebra)

Matrix equation : $Ax = \lambda x$

Trivial solution (regardless of the value of λ)

$$\boxed{x=0}$$

For certain values of λ , called eigenvalues.

For those λ , we set eigen solutions (Nontrivial Solutions)

§ 10.2 / 10.4 : Fourier Series (Trigonometric Series)

Def: The Fourier series representation of a function $f(x)$ ($2L$ -periodic) defined on $[-L, L]$ is

$$\tilde{f}(f) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

whose coefficients are

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx ,$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad \text{and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Recall: Period = 2π

We want $2L = \text{Period}$

If $y = \cos(kx)$

$$\text{Then } 2L = P = \frac{2\pi}{k}$$

$$k = \frac{2\pi}{2L} = \frac{\pi}{L} n,$$

$$n \in \mathbb{Z}^+$$

Recall: 1) let u, v be two functions.

$$\langle u, v \rangle = \int_{-\infty}^{\infty} u(x) v(x) dx = 0$$

if their inner product (dot product) is zero, then

they are orthogonal.

$$2) \int_{-L}^L \cos(x) dx = 2 \int_0^L \cos(x) dx \dots$$

$$3) \int_{-L}^L \sin(x) dx = 0$$

$$4) \int_{-L}^L \cos^2(x) dx = \int_{-L}^L \frac{1}{2}(1 + \cos(2x)) dx \dots$$

Verify the formulas above:

$$\int_{-L}^L f(x) dx = \int_{-L}^L \frac{a_0}{2} dx + \int_{-L}^L \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) dx + \int_{-L}^L \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) dx = 0$$

\downarrow

$$\int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) dx = -\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} \Big|_{-L}^L$$

$$\int_{-L}^L f(x) dx = \frac{a_0}{2} \int_{-L}^L 1 dx = 0 - 0 = 0$$

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n: f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

mult. by $\cos\left(\frac{m\pi x}{L}\right)$ and integrate from $-L$ to L .

$$\int_{-L}^L f(x) \cdot \cos\left(\frac{m\pi x}{L}\right) dx = \int_{-L}^L \frac{a_0}{2} \cdot \cos\left(\frac{m\pi x}{L}\right) dx$$

$$+ \int_{-L}^L \sum_{n=1}^{\infty} a_n \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$+ \int_{-L}^L \sum_{n=1}^{\infty} b_n \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\int_{-L}^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$



$$\int_{-L}^L \cos^2\left(\frac{n\pi x}{L}\right) dx$$

$$\Rightarrow \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = a_n \cdot L, \quad n = 1, 2, 3, \dots$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

~~(*)~~ Derive b_n .

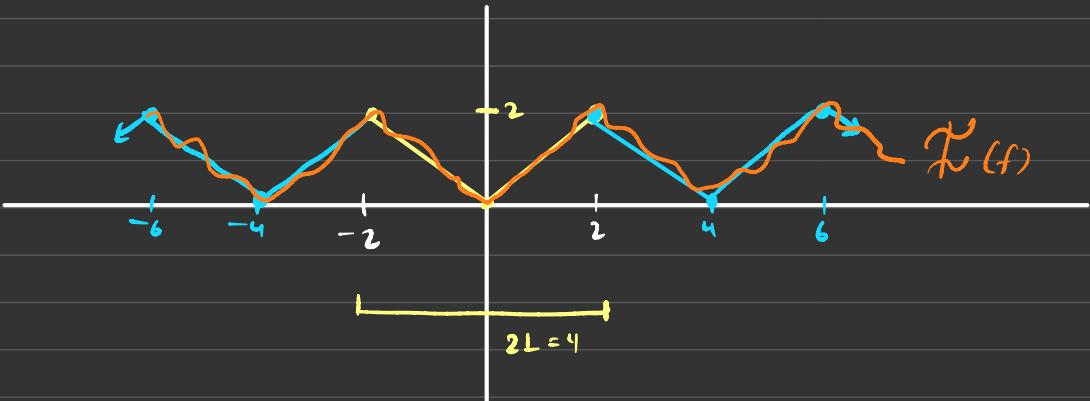
Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \leq x < 0, \\ x, & 0 \leq x < 2; \end{cases}$$

$$f(x+4) = f(x).$$

Determine the coefficients in this Fourier series.

$$\hookrightarrow 2L = 4 \Rightarrow L = 2$$



Find a_0 , a_n and b_n .