

Math 39100 : Feb. 8. 2023 : LECTURE 5

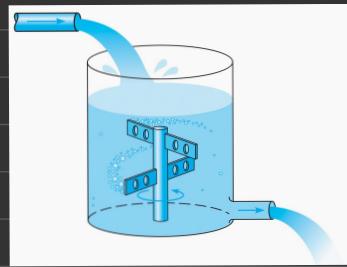
• No class Monday, Feb 13

§ 2.3 Modeling with 1ST ORDER D.E:

(i) Tank problems

(ii) Compound Interest

in



→ out

$Q(t)$:= amount of salt at any time t (grams, kg, lb...)

$\frac{dQ}{dt}$ = rate of change (how fast the amount of salt changing over t)

t = time (minutes, seconds, ...)

$C(t)$ = concentration = $\frac{Q(t)}{V(t)}$ = $\frac{\text{amount}}{\text{Volume}}$ (g/L, ...)

V = volume (liters, gal, ...)

$$\frac{dQ}{dt} = R_{in} - R_{out}$$

$$= r_{in} C_{in} - r_{out} C_{out}$$

$$= h_{in} C_{in} - r_{out} \left(\frac{Q(t)}{V(t)} \right)$$

$$V(t) = V_0 + (r_{in} - r_{out})t$$

\uparrow
initial volume

$$\frac{dQ}{dt} = h_{in} C_{in} - r_{out} \cdot \frac{Q(t)}{V_0 + (r_{in} - r_{out})t}$$

→ linear
D.E.

$$Q(0) = Q_0 \quad (\text{initial salt amount})$$

④ if r_{in} or C_{in} is zero, then it is separable.

Example 1: A tank with a capacity of 100 gallons initially contains 50 gallons of water with 10 pounds of salt in solution. Fresh water enters at a rate of 2 gallons per minute and a well-stirred mixture is pumped out at the same rate. Compute the amount of salt in the tank 10 minutes after the process begins.

$$r_{in} = 2$$

$$C_{in} = 0 \quad (\text{fresh water})$$

$$r_{out} = 2$$

$$C_{out} = \frac{Q(t)}{V(t)} = \frac{Q(t)}{50 + (2-2)t} = \frac{Q(t)}{50}$$

$$Q(0) = 10$$

$$V_0 = 50$$

$$\frac{dQ}{dt} = V_{in} C_{in} - V_{out} C_{out}$$

$$= 2 \cdot 0 - 2 \cdot \frac{Q(t)}{50}$$

$$\boxed{\frac{dQ}{dt} = -\frac{Q(t)}{25}} \quad ; \quad Q(0) = 10$$

$$\int \frac{dQ}{Q(t)} = - \int \frac{1}{25} dt$$

$$\ln(Q(t)) = -\frac{1}{25}t + C$$

$$Q(t) = C e^{-t/25}$$

$$Q(0) = 10 \Rightarrow \textcircled{10 = C}$$

$$Q(t) = 10 e^{-t/25}$$

$$t=10, \quad Q(10) = 10 e^{-10/25} = \boxed{10 e^{-2/5}}$$

A tank initially contains 120L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 2L/min. The well-stirred mixture exits the tank at the same rate.

a) Write a differential equation with a initial condition.

b) Solve for the amount of salt in the tank at any time t .

c) Write an expression for the concentration of salt in the tank at any given time t . $\Rightarrow C(t) = \frac{Q(t)}{V(t)}$

d) Find the limiting amount of salt in the tank as $t \rightarrow \infty$.

$$\text{Given: } r_{in} = 2$$

$$r_{out} = 2$$

$$C_{in} = \gamma$$

$$C_{out} = \frac{Q}{V} = \frac{Q}{120}$$

$$Q(0) = 0 \text{ (pure water)}$$

$$\frac{dQ}{dt} = 2 \cdot \gamma - 2 \cdot \frac{Q}{120}$$

$$\boxed{\frac{dQ}{dt} = 2\gamma - \frac{Q}{60}} , \boxed{Q(0) = 0}$$

$$b) Q(t) = ?$$

$$\frac{dQ}{dt} + \frac{1}{60}Q = 2\gamma$$

$$\mu(t) = e^{\int P(t) dt} \\ = e^{\int \frac{1}{60} dt} = e^{t/60}$$

$$\int \frac{d}{dt} \left[e^{t/60} \cdot Q \right] dt = \int 2\gamma e^{t/60} dt$$

$$\int e^{ax} dx = \frac{e^{ax}}{a}$$

$$\frac{e^{t/60} \cdot Q}{e^{t/60}} = \frac{120\gamma e^{t/60}}{e^{t/60}} + C$$

$$Q(t) = 120\gamma + C e^{-t/60}$$

$$Q(0) = 0 \Rightarrow 0 = 120\gamma + C \Rightarrow C = -120\gamma$$

$$\boxed{Q(t) = 120\gamma - 120\gamma e^{-t/60} \\ = 120\gamma (1 - e^{-t/60})}$$

$$c) C(t) = \frac{Q(t)}{V(t)} = \frac{120\gamma - 120\gamma e^{-t/60}}{120}$$

$$= \boxed{\gamma - \gamma e^{-t/60}}$$

$$a) \lim_{t \rightarrow \infty} Q(t) = \lim_{t \rightarrow \infty} (120\gamma - 120\gamma e^{-t/60})$$

$$= \boxed{120\gamma}$$

Compound Interest : (Finance)

$S(t)$ = amount (value of an investment at any time t)

t = years

$\frac{ds}{dt} = \text{rate of change of the value of the investment.}$

$r = \text{interest rate}$

$k = \text{periodic deposit / withdrawal.}$

$$\frac{ds}{dt} = rs \quad \xrightarrow{\text{OR}}$$

$$\frac{ds}{dt} = rs + k \quad (\text{deposit})$$

$$\int \frac{ds}{s} = \int r dt$$

$$\frac{ds}{dt} = rs - k \quad (\text{withdrawal})$$

$$\ln s = rt + c$$

$$s(t) = ce^{rt}$$

$$s(0) = s_0 \quad (\text{initial condition})$$

some deposit at $t=0$.

$$\frac{ds}{dt} = rs + k \quad (\text{making deposit})$$

$$\frac{ds}{dt} - rs = k$$

$$u(t) = e^{\int r dt} = e^{-rt}$$

$$\int \frac{d}{dt} \left[e^{-rt} \cdot s \right] dt = \int k e^{-rt} dt$$

$$\frac{e^{-rt} \cdot s(t)}{e^{-rt}} = -\frac{k}{r} e^{-rt} + C$$

$$S(t) = -\frac{k}{r} + Ce^{rt}$$

$$S(0) = S_0 \Rightarrow S_0 = -\frac{k}{r} + C \Rightarrow C = S_0 + \frac{k}{r}$$

$$S(t) = -\frac{k}{r} + \left(S_0 + \frac{k}{r}\right)e^{rt}$$

$$S(t) = S_0 e^{rt} + \frac{k}{r} \left(e^{rt} - 1\right)$$

Suppose John is depositing money into a bank account continuously at the rate of \$10,000 per year, and the account earns interest of 4% annually. John began his first year with \$23,000 in the account. Assuming he doesn't make any withdrawals, how much money is in the account after 4 years?

$$\frac{ds}{dt} = 0.04s + 10000$$

$$\frac{ds}{dt} - 0.04s = 10000$$

$$\mu(t) = e^{-\int 0.04 dt} = e^{-0.04t}$$

$$k = 10000$$

$$r = 0.04$$

$$S(0) = 23000$$

$$S(4) = ?$$

$$\int \frac{d}{dt} \left[e^{-0.04t} \cdot S(t) \right] dt = \int 10000 e^{-0.04t} dt$$

$$e^{-0.04t} \cdot S(t) = -\frac{10000}{0.04} e^{-0.04t} + C$$

$$S(t) = -\frac{10000}{0.04} + C e^{0.04t}$$

$$S(0) = 23000 \Rightarrow 23000 = -\frac{10000}{0.04} + C$$

$$C = 23000 + \frac{10000}{0.04}$$

$$\begin{aligned} S(t) &= -\frac{10000}{0.04} + \left(23000 + \frac{10000}{0.04} \right) e^{0.04t} \\ &= \frac{10000}{0.04} \left(e^{0.04t} - 1 \right) + 23000 e^{0.04t} \end{aligned}$$

$$S(4) = 23000 e^{0.16} + \frac{10000}{0.04} \left(e^{0.16} - 1 \right)$$