

Math 39100 : Jan 30, 2023 : LECTURE 2

e.g. Given $u(x, t) = e^{-3t} \sin(2x)$.

Show that $u(x, t)$ is a solution of $u_t = 9u_{xx}$.

$$u_t = -3e^{-3t} \sin(2x)$$

$$u_x = 2\cos(2x) \cdot e^{-3t}$$

$$u_{xx} = -4\sin(2x)e^{-3t}$$

$$-3e^{-3t} \sin(2x) \stackrel{?}{=} 9(-4\sin(2x)e^{-3t})$$

$u(x, t) = e^{-3t} \sin(2x)$ is not a solution to $u_t = 9u_{xx}$.

§2.2 Separable Equations (method 1)

Form :

$$\frac{dy}{dx} = F(x, y)$$

or

$$y' = F(x, y)$$

what is the solution to the D.E. above?

Solution $y(x) = \dots$

(1) If ODE is separable, $f(x) dx + g(y) dy = c$
 $\underbrace{f(x) dx + g(y) dy}$
differentiable form.

(2) Integrate (1)

(3) Express the solution in implicit or explicit form.
 $y = \dots$

Show that the equation

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

is separable, and then find an equation for its integral curves.

→ find the solution to the ODE.

$$(1) \quad \underbrace{(1-y^2)}_{f(y)} dy = \underbrace{x^2}_{g(x)} dx$$

$$f(y) dy = g(x) dx$$

$$(2) \quad \int (1-y^2) dy = \int x^2 dx$$

$$y - \frac{y^3}{3} = \frac{x^3}{3} + C$$

$$3C = D$$

$$3y - y^3 = x^3 + D$$

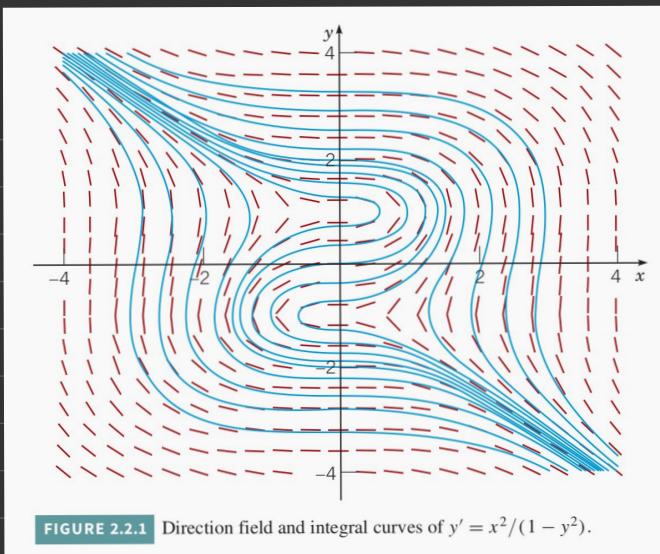


FIGURE 2.2.1 Direction field and integral curves of $y' = x^2 / (1 - y^2)$.

If we impose on initial condition $y(x_0) = y_0$, then we can find a value for C .

Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1, \quad \text{I.C}$$

$$(2y - 2) dy = (3x^2 + 4x + 2) dx$$

$$\int (2y - 2) dy = \int (3x^2 + 4x + 2) dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C \quad \leftarrow \text{General Solution}$$

$$\text{Apply I.C: } (-1)^2 - 2(-1) = 0^3 + 2(0)^2 + 2(0) + C$$

$$3 = C$$

$$\boxed{y^2 - 2y = x^3 + 2x^2 + 2x + 3}$$

a. Find the solution of the given initial value problem in explicit form.

c. Plot the graph of the solution.

c. Determine (at least approximately) the interval in which the solution is defined.

$$dr/d\theta = r^2/\theta, \quad r(1) = 2$$

$$r(\theta) =$$

$$\frac{1}{r^2} dr = \frac{1}{\theta} d\theta$$

$$\int r^{-2} dr = \int \frac{1}{\theta} d\theta$$

$$\int r^{-2} dr = \frac{r^{-1}}{-1}$$

$$-\frac{1}{r} = \ln|\theta| + C$$

$$\frac{1}{r} = -\ln|\theta| + C$$

$$\Rightarrow r = \frac{1}{C - \ln|\theta|}$$

$$r(1) = 2 \Rightarrow 2 = \frac{1}{C - \ln 1}$$

$$2 = \frac{1}{C} \Rightarrow \left(C = \frac{1}{2}\right)$$

$$r(\theta) = \frac{1}{\frac{1}{2} - \ln|\theta|}$$

OR

$$r(\theta) = \frac{1}{\frac{1}{2} - \frac{2 \ln |\theta|}{2}}$$

a)

$$r(\theta) = \frac{2}{1 - 2 \ln |\theta|}$$

$$c) 1 - 2 \ln \theta = 0$$

$$1 = 2 \ln \theta \Rightarrow \frac{1}{2} = \ln \theta \Rightarrow \theta = e^{1/2}$$

$$(0, e^{1/2}) \cup (e^{1/2}, \infty) \rightarrow \text{Domain of } r(\theta)$$

$r(\theta)$ is defined in $(0, e^{1/2})$

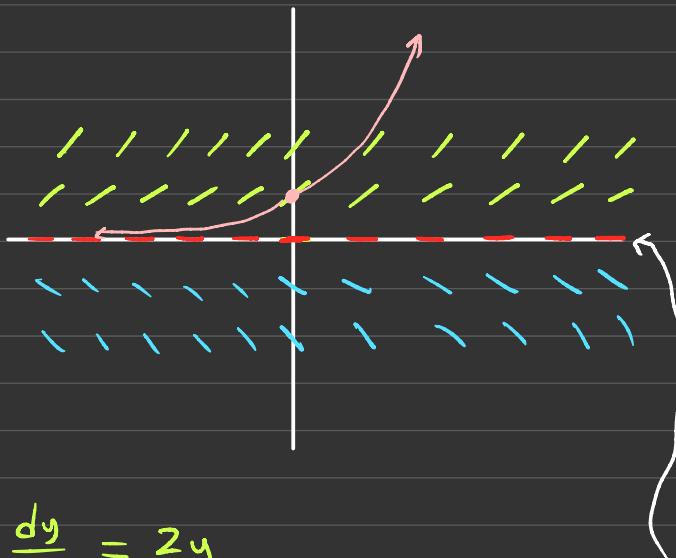
Sketch slope field

$$\frac{dy}{dx} = m = \text{slope}$$

/ + slope | undefined

\ - slope — Slope is zero

eg. Draw a slope field of $\frac{dy}{dx} = 2y$.



$$y=0, \frac{dy}{dx}=0$$

$$y=1, \frac{dy}{dx}=2(1)=2$$

$$y=2, \frac{dy}{dx}=2(2)=4$$

$$y=-1, \frac{dy}{dx}=2(-1)=-2$$

$$\frac{dy}{dx} = 2y$$

Equilibrium solution

$$\int \frac{1}{y} \cdot dy = \int 2 dx$$

$$\ln|y| = 2x + C$$

$$y = e^{2x+C}$$

$$y = C e^{2x} \quad \text{use } (0,1) \text{ to find } C.$$

$$y = e^{2x}$$

Method 2: Homogeneous 1st order D.E.:

IF WE HAVE $y' = f(x, y)$ AND CAN BE

WRITTEN IN THE FORM $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

$$\frac{dy}{dx} = \frac{4y - 3x}{2x - y}$$

divide every term by x
on the right side.

$$\frac{dy}{dx} = \frac{4\left(\frac{y}{x}\right) - 3}{2 - \left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right) \quad (\textcircled{K})$$

Let $\boxed{v} = \frac{y}{x} \Rightarrow y = vx$ where v is a function of x .

$$\boxed{\frac{dy}{dx}} = v + x \cdot \frac{dv}{dx}$$

Sub in v and $\frac{dy}{dx}$ into eq. \textcircled{K} So that it is
a separable eq.

$$v + x \cdot \frac{dv}{dx} = \frac{4v - 3}{2 - v}$$

$$x \cdot \frac{dv}{dx} = \frac{4v - 3}{2 - v} - v$$

$$x \frac{dv}{dx} = \frac{4v - 3 - v\widehat{(2-v)}}{2-v}$$

$$x \frac{dv}{dx} = \frac{4v - 3 - 2v + v^2}{2-v}$$

$$x \frac{dv}{dx} = \frac{v^2 + 2v - 3}{2-v} \quad \left(\begin{array}{l} \text{this} \\ \text{is} \\ \text{separable} \end{array} \right)$$

$$\frac{2-v}{v^2+2v-3} dv = \frac{1}{x} dx$$

$$\int \frac{2-v}{v^2+2v-3} dv = \int \frac{1}{x} dx$$

\nearrow
partial fractions

$$= \ln|x| + C$$

$$\int \frac{2-v}{(v+3)(v-1)} dv = \int \frac{-5/4}{v+3} dv + \int \frac{1/4}{v-1} dv$$

$$= -5/4 \ln|v+3| + \frac{1}{4} \ln|v-1|$$

$$\frac{2-v}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1}$$

$$2-v = A(v-1) + B(v+3)$$

$$\text{let } v=1 : 1 = B(1+3)$$

$$B = \frac{1}{4}$$

$$v=-3 : 5 = A(-3-1)$$

$$-\frac{5}{4} = A$$

To be continued ...