

Math 39100 : April. 19. 2023 : LECTURE 20

Quiz 5: Monday, April 24 (on Sections 4.3 and 5.2)

Exam 2: Monday, May 1 (starting from 3.5, ...)

Exam 3: (optional) Friday, May 12

Time: 4 - 5:15 pm

Room: Nac 1/511

FINAL EXAM: May, 18 from 3:30 - 5:45 pm

Room: TBA

§5.5 cont. from last time:

$$2 \cdot (r-1)(r) a_0 x^r - r a_0 x^r + a_0 x^r$$

$$+ \sum_{k=1}^{\infty} [z(k+r-1)(k+r)a_k - (k+r)a_k + a_k + a_{k-1}] x^{k+r} = 0$$

$$x^r a_0 (zr(r-1) - r + 1)$$

$$+ \sum_{k=1}^{\infty} [z(k+r-1)(k+r)a_k - (k+r)a_k + a_k + a_{k-1}] x^{k+r} = 0$$

$$\begin{aligned} zr(r-1) - r + 1 &= 0 \\ r = 1, 1/2 & \quad \left| \begin{array}{l} 2(k+r-1)(k+r)a_k - (k+r)a_k + a_k + a_{k-1} = 0 \end{array} \right. \end{aligned}$$

↗

indicial equation
 (same equation by
 solving corresponding
 Euler eq.)

$$a_k \left(2(k+r-1)(k+r) - (k+r) + 1 \right) = -a_{k-1}$$

$$a_k = \frac{-a_{k-1}}{2(k+r-1)(k+r) - (k+r) + 1}, \quad k=1, 2, \dots$$

$$\text{using } r=1, \quad a_k = \frac{-a_{k-1}}{2k(k+1)-k} = \frac{-a_{k-1}}{2k^2+k}$$

$$k=1, \quad a_1 = \boxed{\frac{-a_0}{3}}$$

$$k=2 : \quad \boxed{a_2} = -\frac{a_1}{10} = \boxed{\frac{a_0}{30}}$$

$$\text{So, } y_1 = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$(r=1)$$

$$= x^1 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \right)$$

$$\boxed{y_1 = x \left(a_0 - \frac{a_0}{3} x + \frac{a_0}{30} x^2 + \dots \right)}$$

$$\text{for } r=\frac{1}{2} : \quad a_k = -\frac{a_{k-1}}{2(k-\frac{1}{2}) - k + \frac{1}{2}}$$

$$= -\frac{a_{k-1}}{(2k-1)(k+\frac{1}{2}) - k + \frac{1}{2}}$$

$$= - \frac{a_{k-1}}{2k^2 - \frac{1}{2} - k + \frac{1}{2}}$$

$$a_k = - \frac{a_{k-1}}{2k^2 - k} \quad k=1, 2, \dots$$

$$k=1, \quad a_1 = -a_0$$

$$k=2 : \quad a_2 = -\frac{a_1}{6} = \frac{a_0}{6}$$

⋮

$$r=\frac{1}{2}$$

$$y_2 = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$= x^{1/2} (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= x^{1/2} (a_0 - a_0 x + \frac{a_0}{6} x^2 + \dots)$$

$$\text{General Soln: } y = C_1 y_1 + C_2 y_2$$

6.1 and 6.2: The Laplace Transform

Find the solution of the differential equation

$$y'' - y' - 2y = 0$$

that satisfies the initial conditions

$$y(0) = 1, \quad y'(0) = 0.$$

(6.2)

coming up

Def:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

a.k.a. Integral Transform

Find $\mathcal{L}\{1\}$.

$$F(s) = \int_0^{\infty} e^{-st} \cdot 1 \, dt = \frac{-e^{-st}}{-s} \Big|_0^{\infty} = \lim_{b \rightarrow \infty} \frac{-e^{-sb}}{-s} \Big|_0^b$$

$$= \lim_{b \rightarrow \infty} \left[\underbrace{\frac{-e^{-sb}}{-s}}_0 + \frac{1}{s} \right]$$

$s > 0$

$$= \boxed{\frac{1}{s}}, \quad s > 0$$

$\mathcal{L}\{k\}$ where k is any constant.

$$\boxed{\mathcal{L}\{k\} = \frac{k}{s}}$$

Find $\mathcal{L}\{e^{at}\}$.

Using the table:

$$\mathcal{L}\{e^{st}\} = \frac{1}{s-s}$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot \underbrace{e^{at}}_{f(t)} dt$$

$$= \int_0^\infty e^{(a-s)t} dt$$

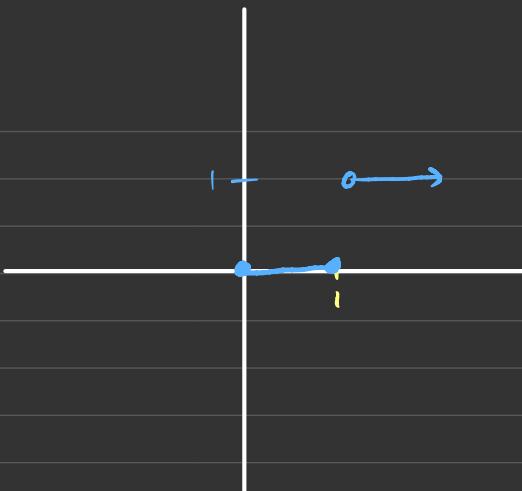
$$= \frac{e^{(a-s)t}}{a-s} \Big|_0^\infty$$

$$= 0 - \frac{1}{a-s} \quad s > a$$

$$= \boxed{\frac{1}{s-a}}$$

$$\text{let } f(t) = \begin{cases} 0, & t \leq 1 \\ 1, & t > 1 \end{cases}$$

Try $\mathcal{L}\{f(t)\}$.



* Laplace Transform is a linear Operator

$$\frac{d}{dx} [cf + g] = c \frac{d}{dx}[f] + \frac{d}{dx}[g]$$

$$\mathcal{L}\{c \cdot f(t) + g(t)\} = c \cdot \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

6.2 Solving 2nd order linear D.E with constant

Coeff. with L.T.

$\mathcal{L}\{y'\}$ and $\mathcal{L}\{y''\}$?

$$\mathcal{L}\{y'\} = \int_0^\infty e^{-st} \cdot y' dt \quad \xrightarrow{\text{Integration by Parts}}$$

$$u = e^{-st} \quad v = y$$

$$du = -se^{-st} \quad dv = y'$$

$$= e^{-st} y \Big|_0^\infty + s \int_0^\infty e^{-st} y dt$$

$$= 0 - y(0) + s \underbrace{\int_0^\infty e^{-st} y dt}_{\mathcal{L}\{y\}}$$

$$\boxed{\mathcal{L}\{y'\} = -y(0) + s \mathcal{L}\{y\}} \quad \text{⊕ Remember this!}$$

Find $\mathcal{L}\{y''\}$.

$$\mathcal{L}\{y''\} = \int_0^\infty e^{-st} \cdot y'' dt \quad \xrightarrow{\text{I.B.P.}}$$

$$u = e^{-st} \quad v = y'$$

$$du = -se^{-st} \quad dv = y''$$

$$= e^{-st} \cdot y' \Big|_0^\infty + s \underbrace{\int_0^\infty e^{-st} \cdot y' dt}_{\text{def. } \mathcal{L}\{y'\}}$$

$$= 0 - y'(0) + s \mathcal{L}\{y'\}$$

$$= -y'(0) + s(-y(0) + s \mathcal{L}\{y\})$$

$$\boxed{\mathcal{L}\{y''\}} = \boxed{-y'(0) - sy(0) + s^2 \mathcal{L}\{y\}}$$