Math 39100 : Mar. 27. 2023 : LECTURE 16

Quiz 4: Monday, April 3 (on 3.6, and 3.7 / 3.8)

Exam 2: Monday, May 1

Exam 3: (optional) Friday, May 12

CH5 : Series Solutions of D.E :

§ 5.1 Review of Power Series

Def: A power series is a series of the form $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ Centered at $x = x_0$.

Notation:

ending point

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

infinite sum

starting point

Finite Sum :

$$\sum_{n=0}^{k} a_n x^n = a_0 + a_1 x^1 + a_2 x^2 + \dots + a_k x^k$$

Centered at $x_0 = 0 \implies \sum_{n=0}^{\infty} a_n (x-0)^n = \sum_{n=0}^{\infty} a_n x^n$

Properties:

(i) $\sum_{n=0}^{\infty} a_n \pm \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} [a_n \pm b_n]$

(ii) If $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

(*) Most common test to test for absolutely convergence of a power series is the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

(i) if $L < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

(ii) if $L > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges.

(iii) if $L = 1$, then the test fails (inconclusive)

This gives us two things : I.O.C and R.O.C.

Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n2^n}.$$

$a_n = \frac{(x+1)^n}{n\,2^n}$

$a_{n+1} = \frac{(x+1)^{n+1}}{(n+1)\,2^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{(n+1)\,2^{n+1}}\cdot\frac{n\cdot 2^n}{(x+1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x+1)^n(x+1)}{(n+1)\cdot 2^n\cdot 2}\cdot\frac{n\cdot 2^n}{(x+1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x+1)\cdot n}{2(n+1)}\right|$$

$$= |x+1|\cdot\underbrace{\lim_{n\to\infty}\left|\frac{n}{2(n+1)}\right|}$$

$$= |x+1|\cdot\frac{1}{2} < 1$$

$$\frac{1}{2}|x+1| < 1$$

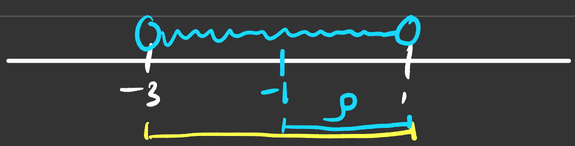
$$|x+1| < 2$$

$$-2 < x+1 < 2$$

$$\boxed{-3 < x < 1}$$

I.O.C: $(-3, 1)$

(feel free to test the end points



R.O.C: $|-^-3| = 4$

Radius of Convergence is denoted by $\rho = \frac{4}{2} = 2$

(iii) Shifting indices of a summation:

$$\sum_{n=0}^{\infty} a_n = \sum_{k=0}^{\infty} a_k \qquad n, k \text{ are dummy}$$

Write $\sum_{n=2}^{\infty} a_n x^n$ as a series whose first term corresponds to $n = 0$ rather than $n = 2$.

let $k = n-2 \Rightarrow n = k+2$

$n=2, \quad k=0$

$n=\infty, \quad k=\infty$

$$\sum_{k=0}^{\infty} a_{k+2} X^{k+2}$$

(iv) let's suppose f is a differentiable function.

then

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

$$f'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$f''(x) = 2a_2 + 3\cdot 2\, a_3(x-x_0) + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

(ii) A function can be approximated by using Taylor Series:

$$f(x) = \sum_{n=0}^{\infty} \boxed{a_n} (x - x_0)^n$$

where $\boxed{a_n = \frac{f^{(n)}(x_0)}{n!}}$

5.2 Series Solutions near an Ordinary Point, part I.

Consider Diff. equation of the form:

$$\boxed{P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0}$$

where $P(x)$, $Q(x)$ and $R(x)$ are functions of the indep. variable x.

def: A point x_0 such that $P(x_0) \neq 0$ is called an **ordinary point**.

We are going assume the solution form is

$$y = \sum_{n=0}^{\infty} a_n (x - \underset{\substack{\uparrow \\ 0}}{x_0})^n$$

for simplicity assume $x_0 = 0$

$$\Rightarrow \boxed{y = \sum_{n=0}^{\infty} a_n x^n} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Goal is to find these coeff. $a_0, a_1, a_2, a_3, \ldots$

Find a series solution of the equation $[x_0 = 0]$

$$y'' + y = 0, \quad -\infty < x < \infty.$$

Assume $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution.

Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

from ch3:

$y'' + y = 0$

$r^2 + 1 = 0$

$r = \pm i$

$y = c_1 \cos(x) + c_2 \sin(x)$

Now, plug in y and y'' into the given eq:

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$k = n-2$
$\Rightarrow n = k+2$

$k = n$

$\boxed{\text{* Powers to } k}$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} + a_k\right] x^k = 0$$

$$\Rightarrow (k+2)(k+1) a_{k+2} + a_k = 0$$

④ Solve for the highest subscript

$$\boxed{a_{k+2} = \frac{-a_k}{(k+2)(k+1)}} \quad k = 0, 1, 2, \ldots$$

Recurrence formula (formula for the coefficients)

$k=0$, $\quad a_2 = -\frac{a_0}{2\cdot 1} = -\frac{a_0}{2}$

$k=1$, $\quad a_3 = -\frac{a_1}{3\cdot 2} = -\frac{a_1}{6}$

$k=2$, $\quad a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{4\cdot 3\cdot 2}$

$k=3$, $\quad a_5 = -\frac{a_3}{5\cdot 4} = \frac{a_1}{6\cdot 5\cdot 4}$

Solution : $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$

$$= a_0 + a_1 x - \frac{a_0}{2} x^2 - \frac{a_1}{6} x^3 + \frac{a_0}{4\cdot 3\cdot 2} x^4 + \frac{a_1}{6\cdot 5\cdot 4} x^5 + \ldots$$

$$y = a_0 \Big(\overbrace{1 - \frac{1}{2} x^2 + \frac{1}{4\cdot 3\cdot 2} x^4 + \ldots}^{y_1} \Big)$$

$$+ a_1 \Big(\overbrace{x - \frac{1}{6} x^3 + \frac{1}{6\cdot 5\cdot 4} x^5 + \ldots}^{y_2} \Big)$$

$$\boxed{y = a_0 \cos(x) + a_1 \sin(x)}$$

where $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

and $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$