

# Math 39100 : April. 3 . 2023 : LECTURE 18

Exam 2: Monday, May 1 (Starting from 3.5, ... )

Exam 3: (optional) Friday, May 12

Time: 4 - 5:15 pm

Room: TBA

FINAL EXAM: May , 18 from 3:30 - 5:45 pm

Room: TBA

Cont. from last time:

Solve  $y'' + \sin(x)y' + \cos(x)y = 0$ ,  $y(0) = a_0$  and  
 $y'(0) = a_1$ .

Assume  $x_0 = 0$ .

$$\text{Solu: } y = \sum_{n=0}^{\infty} a_n x^n$$

$$\boxed{\text{Recall: } a_n = \frac{y^{(n)}(x_0)}{n!}}$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$a_2 : y'' = -\sin(x)y' - \cos(x)y \Big|_{x_0=0}$$

$$= -\sin(\sigma) y'(\sigma) - \cos(\sigma) y(\sigma)$$

$$y''(0) = -y(0) = -a_0$$

$$a_2 = -\frac{a_0}{2!}$$

$$a_3 : y''' = -\cos(x)y' - \sin(x)y'' + \sin(x)y - \cos(x)y' \Big|_{x_0=0}$$

$$y'''(0) = -\cos(0)y'(0) - \sin(0)y''(0) + \overset{0}{\sin(0)y(0)} - \overset{0}{\cos(0)y'(0)}$$

$$= -y'(0) - y(0)$$

$$y'''(0) = -2y'(0) = -2a_1$$

$$a_3 = -\frac{2a_1}{3!}$$

$$\text{Solu: } y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{2}{3!} a_1 x^3 + \dots$$

## § 5.4 Euler Equations ; Regular Singular Points

$$\text{Form: } a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = 0$$

$$\text{Interest: } \alpha x^2 y'' + \beta x y' + \gamma y = 0 \leftarrow \text{Euler Eq.}$$

Assume the soln form  $y = x^r$  .  $x > 0$

Goal: Find what is/are  $r$ ?

$$y' = rx^{r-1} \quad \text{and} \quad y'' = r(r-1)x^{r-2}$$

So then,  $\alpha x^2 r(r-1)x^{r-2} + \beta x \cdot r x^{r-1} + \gamma x^r = 0$

$$\alpha r(r-1)x^r + \beta r x^r + \gamma x^r = 0$$

$$x^r(\alpha r(r-1) + \beta r + \gamma) = 0$$

$$\alpha r(r-1) + \beta r + \gamma = 0 \quad \leftarrow \text{indicial equation}$$

(Quadratic eq.)

3 cases to consider for the roots of the eq. above:

i) Real distinct roots :  $r_1 \neq r_2$  :  $y = C_1 x^{r_1} + C_2 x^{r_2}$

ii) Real, repeated :  $r_1 = r_2$  :  $y = C_1 x^{r_1} + C_2 x^{r_1} \ln x$

iii) Complex :  $r = a + bi$  :  $y = C_1 x^a \cos(b \ln x) + C_2 x^a \sin(b \ln x)$

$$\begin{aligned} x^{a+bi} &= e^{\ln x^{a+bi}} = e^{(a+bi)\ln x} \\ &= e^{a\ln x} \cdot e^{b\ln x i} \\ &= x^a \cdot e^{(b\ln x)i} \end{aligned}$$

$$= x^a (\cos(b \ln x) + i \sin(b \ln x))$$

Solve

$$2x^2y'' + 3xy' - y = 0, \quad x > 0.$$

Assume the soln form is  $y = x^r$ .

$$\left. \begin{array}{l} \alpha = 2 \\ \beta = 3 \\ \gamma = -1 \end{array} \right\} \begin{array}{l} \alpha r(r-1) + \beta r + \gamma = 0 \\ 2r(r-1) + 3r - 1 = 0 \end{array}$$

$$2r^2 + r - 1 = 0$$

$$(2r-1)(r+1) = 0$$

$$r_1 = \frac{1}{2}, \quad r_2 = -1$$

General soln:

$$y = C_1 |x|^{\frac{1}{2}} + C_2 |x|^{-1}$$

Since  $x > 0$ ,

$$y = C_1 x^{\frac{1}{2}} + C_2 x^{-1}$$

Solve

$$x^2y'' + 5xy' + 4y = 0, \quad x > 0.$$

$$\left. \begin{array}{l} \alpha = 1 \\ \beta = 5 \\ \gamma = 4 \end{array} \right\} \begin{array}{l} r(r-1) + 5r + 4 = 0 \\ r^2 + 4r + 4 = 0 \end{array}$$

$$(r+2)^2 = 0$$

$$r_1 = r_2 = -2$$

real, repeated

Solu: 
$$y = C_1 \bar{x}^{-2} + C_2 \bar{x}^{-2} \cdot \ln x$$

Solve

$$x^2 y'' + xy' + y = 0. \quad x > 0.$$

$$\alpha = 1$$

$$r(r-1) + r + 1 = 0$$

$$\beta = 1$$

$$r^2 + 1 = 0$$

$$r = \pm i = 0 \pm i$$

Solu: 
$$y = C_1 \cos(\ln(x)) + C_2 \sin(\ln x)$$

Singular Point(s): Suppose we have

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

We say  $x_0$  is a singular point if  $P(x_0) = 0$ .

Regular vs. Irregular Singular points.

To be continued...