

Math 39100 : Feb. 6. 2023 : LECTURE 4

So far : (i) If an ODE is separable i.e. $f(x) dx + g(y) dy = 0$

(ii) If an ODE is Homogeneous, then

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \dots$$

(iii) If an ODE is linear : $y' + p(x)y = q(x)$

use the method of Integrating Factor

$$\mu(x) = e^{\int p(x) dx}$$

⋮

§ 2.6 EXACT EQUATIONS

Motivation : Given $2x + y^2 + 2xyy' = 0$.

the solution is $\phi(x, y) = x^2 + xy^2$.

rewritten

$$2x + y^2 + 2xy \frac{dy}{dx} = 0$$

$$\underbrace{(2x + y^2)}_{\phi_x} dx + \underbrace{2xy}_{\phi_y} dy = 0$$

$$\phi_x(x, y) = 2x + y^2$$

$$\phi_y(x, y) = 2xy$$

We have

$$\int \frac{\partial}{\partial x} \left[\phi(x, y) \right] = \int 0 \, dx$$

$$\phi(x, y) = C$$

Thus,

$$\boxed{\text{Solution : } x^2 + xy^2 = C}$$

Above is true if the given D.E is exact.

$$\underbrace{(2x + y^2)}_{\phi_x = M} \, dx + \underbrace{2xy \, dy}_{\phi_y = N} = 0$$

check: ✓

$$M_y = 2y$$

$$\underline{\text{Eq. Form : }} M \, dx + N \, dy = 0$$

$$N_x = 2y$$

Test $M_y = N_x$, then the ODE is exact.

Here is the method :

$$\phi(x, y) = \int M \, dx$$

$$= \int (2x + y^2) \, dx$$

$$\phi(x, y) = x^2 + y^2 x + g(y)$$

$$N = \phi_y = 2yx + g'(y)$$

$$2xy = 2xy + g'(y)$$

$$\int 0 \, dy = \int g'(y) \, dy$$

$$C_1 = g(y)$$

$$\text{Thus, } \phi(x, y) = x^2 + xy^2 + C_1$$

$$\Rightarrow C = x^2 + xy^2$$

Theorem 2.6.1

Let the functions M, N, M_y , and N_x , where subscripts denote partial derivatives, be continuous in the rectangular¹⁹ region $R: \alpha < x < \beta, \gamma < y < \delta$. Then equation (6)

$$M(x, y) + N(x, y)y' = 0 \Rightarrow M \, dx + N \, dy = 0$$

is an exact differential equation in R if and only if

$$M_y(x, y) = N_x(x, y) \quad (10)$$

at each point of R . That is, there exists a function ψ satisfying equations (7),

$$\psi_x(x, y) = M(x, y), \quad \psi_y(x, y) = N(x, y),$$

if and only if M and N satisfy equation (10).

Solve the differential equation

$$(y \cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0.$$
$$\underbrace{(y \cos x + 2xe^y)}_M \, dx + \underbrace{(\sin x + x^2e^y - 1)}_N \, dy = 0$$

Show that it is exact.

$$\begin{aligned} M_y &= \cos(x) + 2x e^y \\ N_x &= \cos(x) + 2x e^y \end{aligned} \quad] \text{ since } M_y = N_x, \text{ it is exact.}$$

$$\phi(x, y) = \int M \, dx$$

$$= \int (y \cos(x) + 2x e^y) \, dx$$

$$= y \sin(x) + x^2 e^y + g(y) \leftarrow$$

$$\phi_y = \sin(x) + x^2 e^y + g'(y) = N$$

$$\sin(x) + x^2 e^y + g'(y) = \sin x + x^2 e^y - 1$$

$$g'(y) = -1$$

$$\Rightarrow g(y) = -y$$

final answer: $C = y \sin(x) + x^2 e^y - y$

Find a particular solution if $y(\pi) = 0$.

$$C = 0 \cdot \sin(\pi) + \pi^2 e^0 - 0$$

$$C = \pi^2$$

$$\pi^2 = y \sin(x) + x^2 e^y - y$$

Solve $(3xy + y^2) + (x^2 + xy)y' = 0$.

$$\underbrace{(3xy + y^2)}_M dx + \underbrace{(x^2 + xy)}_N dy = 0$$

$$M_y = 3x + 2y \quad N_x = 2x + y$$

Not exact since $M_y \neq N_x$.

Solve $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$.

$$\underbrace{(3x^2 - 2xy + 2)}_M dx + \underbrace{(6y^2 - x^2 + 3)}_N dy = 0$$

$$M_y = -2x \quad N_x = -2x$$

Exact ☺

$$\phi(x, y) = \int N dy$$

$$= \int (6y^2 - x^2 + 3) dy$$

$$\phi(x, y) = 2y^3 - x^2y + 3y + g(x)$$

$$\phi_x = -2xy + g'(x) = M$$

$$-2xy + g'(x) = 3x^2 - 2xy + 2$$

$$g'(x) = 3x^2 + 2$$

$$\int g'(x) \, dx = \int (3x^2 + 2) \, dx$$

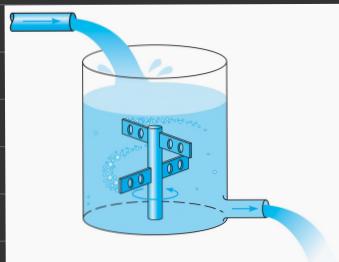
$$g(x) = x^3 + 2x$$

$$C = 2y^3 - x^2y + 3y + x^3 + 2x$$

§ 2.3 Modeling with 1st order D.E

(i) Tank problems

(ii) Exponential Growth / Decay



To be continued ...