1. (10 points) Solve the initial value problem: $e^{x+y} + \sin(y) + (x\cos(y) + e^{x+y} - e^y)y' = 0,\ y(0) = e.$

$$(e^{x+y} + \sin y)dx + (x\cos y + e^{x+y} - e^y)dy = 0$$

$$M_y = e^{x+y} + \cos y \qquad N_x = \cos y + e^{x+y}$$

$M_y = N_x$ ✓ Exact ✓

✓

$$\int x\cos y + e^{x+y} - e^y\, dy = x\sin y + e^{x+y} - e^y + g(x)$$

$$\sin y + e^{x+y} + g'(x) = e^{x+y} + \sin y$$

$$g'(x) = 0$$

$$g(x) = c$$

$$c = x\sin y + e^{x+y} - e^y \qquad y(0) = e$$

$$c = 0\sin e + e^e - e^e$$

$$c = 0$$ ✓

$$\boxed{0 = x\sin y + e^{x+y} - e^y}$$ ✓

2. **(10 points)** A tank initially contains 200 gallons of brine whose salt concentration is 3 lb/gal. Brine whose salt concentration is 2 lb/gal flows into the tank at the rate of 4 gal/min. The mixture flows out at the same rate. Find the salt content of the brine at the end of 20 min.

$V_0 = 200$ gal $\quad \frac{200\,\text{gal}}{1}\,\frac{3\,\text{lb}}{1\,\text{gal}} = 600\,\text{lb} = Q(0)$

ok

$Q(0) = 600$ lb

$r_{in} = 4$ gal/min $= r_{out}$ $\quad \frac{dQ}{dt} = r_{in}c_{in} - r_{out}\frac{Q(t)}{V_0 + (r_{in} - r_{out})t}$

$c_{in} = 2$ lb/gal

$$\frac{dQ}{dt} = 4(2) - \frac{4Q(t)}{200}$$

$$\frac{dQ}{dt} = 8 - \frac{Q}{50}$$

$Q' + \frac{1}{50}Q = 8 \qquad \mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{1}{50}dt}$

$\mu(t) = e^{t/50}$

$Q e^{t/50} = \int 8e^{t/50}\,dt$

$Q e^{t/50} = 400e^{t/50} + c, \quad Q(0) = 600$

$600 = 400 + c$

$c = 200$

$$Q = 400 + \frac{200}{e^{t/50}}$$

$$Q(20) = 400 + \frac{200}{e^{2/5}}\ \text{lbs}$$

At the end of 20 min, the salt content of brine will be $400 + \frac{200}{e^{2/5}}$ pounds.

3. (10 points) Identify the differential equation that has complex roots of its characteristic polynomial and find its general solution.

I. $y'' - y' = 0$.

$r^2 - r = 0$

$r(r-1) = 0$

II. $4y'' - 5y = 0$.

$4r^2 - 5r = 0$

$r(4r-5) = 0$

III. $y'' + 4y' + 5y = 0$.

$r^2 + 4r + 5 = 0$

$r = \dfrac{-4 \pm \sqrt{16 - 4(1)(5)}}{2}$

$r = \overset{\alpha}{-2} \pm \underbrace{i}_{\beta}$

$$\boxed{y = e^{-2t}(c_1 \cos t + c_2 \sin t)}$$

4. For the equation $x^2y'' - 7xy' + 16y = 0$, $x > 0$, $y_1 = x^4$ is a solution.

(a) (10 points) Use the method of Reduction of Order to obtain a second, independent solution.

(b) (10 points) Compute the Wronskian of the pair of solutions

a) Assume $y = vy_1 = vx^4$.

Then $y' = v'x^4 + 4x^3v$ and $y'' = v''x^4 + 8x^3v' + 12x^2v$.

So, $x^2(v''x^4 + 8x^3v' + 12x^2v) - 7x(v'x^4 + 4x^3v) + 16vx^4 = 0$

$$x^6v'' + 8x^5v' + 12x^4v - 7x^5v' - 28x^5v + 16vx^4 = 0$$

$$x^6v'' + x^5v' = 0$$

$$\Rightarrow v'' + \frac{1}{x}v' = 0$$

let $w = v'$
$w' = v''$

$$\Rightarrow w' + \frac{1}{x}w = 0$$

$$\int \frac{dw}{w} = -\int \frac{1}{x}\,dx$$

$$\ln(w) = -\ln(x) + C_1$$

$$w = C_1x^{-1}$$

$$\int dv = \int C_1x^{-1}\,dx$$

$$v = C\ln(x) + C_2$$

Thus, $y = vy_1 = (C\ln(x) + C_2)x^4$

$$= C\cdot x^4\ln x + C_2x^4.$$

Therefore, $\boxed{y_2 = x^4\ln x}$

(b) $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} x^4 & x^4\ln x \\ 4x^3 & 4x^3\ln x + x^3 \end{vmatrix}$$

$$= 4x^7\ln x + x^7 - 4x^7\ln x$$

$$= \boxed{x^7}$$

5. (10 points) Which of the following represents the Wronskian of the pair of two linearly independent solutions of the given differential equation:

$$t^2y'' - y' + 6ty = 0, \ t > 0$$

$y'' - \frac{1}{t^2}y' + \frac{6}{t} = 0$

a) $W(y_1, y_2) = Ce^t$

b) $W(y_1, y_2) = Ct^{-t}$

c) $W(y_1, y_2) = Ce^{t^2}$

d) $W(y_1, y_2) = Ce^{1/t}$

e) none of the above.

$W = ce^{-\int p(t)dt} = ce^{\int \frac{1}{t^2}dt} = \boxed{ce^{-1/t}}$

$W = ce^{1/t}$

$t^{-2} \to -1t^{-1}$

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6. (10 points) Consider the first-order linear differential equation $ty' + 3y = 4e^t$. Which of the following is a suitable integrating factor that could be used to solve this equation?

a) $\mu(t) = e^{3t}$

b) $\mu(t) = t^3$

c) $\mu(t) = 3t$

d) $\mu(t) = t^{-1}$

e) $\mu(t) = t^{-3}$

$y' + \frac{3}{t}y = \frac{4e^t}{t}$

$\mu(t) = e^{\int 3/t\, dt} = e^{3\ln t} = t^3$

7. (10 points) The differential equation $(1+t)\frac{d^3y}{dt^3} - \sin(t)\frac{d^2y}{dt^2} = t^2 - 4y$ is classified as...

~~a) Linear, 2^{nd} order, ordinary differential equation.~~

b) Linear, 3^{nd} order, ordinary differential equation.

~~c) Nonlinear, 3^{nd} order, ordinary differential equation.~~

~~d) Nonlinear, 2^{nd} order, partial differential equation.~~

e) Linear, 3^{nd} order, partial differential equation.

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8. (10 points) Which of the following is a solution to the differential equation $2y'' - y' - 1 = 0$. Assume that C_1 and C_2 are two arbitrary constants.

~~a) $y(t) = C_1 \cos(t/2) + C_2 \sin(t)$~~

b) $y(t) = C_1 e^{-t/2} + C_2 t e^{-t/2}$

c) $y(t) = C_1 e^{-t/2} + C_2 e^{t}$

d) $y(t) = C_1 e^{2t} + C_2 e^{t}$

e) none of the above.

$2r^2 - r - 1 = 0$

$r = \frac{1 \pm \sqrt{1 - 4(2)(-1)}}{4}$

$r = \frac{1}{4} \pm \frac{3}{4}$

$r_1 = 1 \quad r_2 = \frac{-1}{2}$

I made a typo here.