

Math 39100 : Feb. 15. 2023 : LECTURE 6

- No class on Monday, Feb 20.
- We have class on Tuesday, Feb 21 (Monday's schedule)
- Quiz 2: Tuesday, Feb 21. (2.3 and 2.6)

§ 2.3 cont.

Suppose that one opens an individual account at age 25 and makes annual investments of \$2000 thereafter in a continuous manner. Assuming a rate of return of 8%, what will be the balance in the account at age 65?

$$S(t) = -\frac{k}{r} + \left(\frac{k}{r} + S_0 \right) e^{rt} \quad \frac{dS}{dt} = rS + k$$

$$S(40) = -\frac{2000}{0.08} + \left(\frac{2000}{0.08} \right) e^{(0.08)(40)}$$

$$S(0) = 0$$

$$k = 2000$$

$$r = 0.08$$

$$S(40) = ?$$

§ 2.9 Review (Miscellaneous problems) and

Reduction of order

Separable, linear Eq (Integration factor), Homogeneous and Exact

In each of Problems 1 through 24, solve the given differential equation.
If an initial condition is given, also find the solution that satisfies it.

1. $\frac{dy}{dx} = \frac{x^3 - 2y}{x}$

2. $\frac{dy}{dx} = \frac{1 + \cos x}{2 - \sin y}$, Separable

3. $\frac{dy}{dx} = \frac{2x + y}{3 + 3y^2 - x}$, $y(0) = 0$

4. $\frac{dy}{dx} = 3 - 6x + y - 2xy$

$$(3 + 3y^2 - x) dy = (2x + y) dx$$

$$\Rightarrow \underbrace{(3 + 3y^2 - x)}_N dy + \underbrace{(2x + y)}_M dx = 0$$

1. $\frac{dy}{dx} = \frac{x^3}{x} - \frac{2y}{x}$

$M_y = N_x$

$-1 \stackrel{?}{=} -1 \quad \checkmark$

\Rightarrow Exact

$$\frac{dy}{dx} + \frac{2}{x} y = x^2 \quad \leftarrow \text{linear D.E.}$$

;

4. $\frac{dy}{dx} = 3 - 6x + y - 2xy$

$$\frac{dy}{dx} = 3 - 6x + y(1 - 2x)$$

$$\frac{dy}{dx} + \underbrace{(1 - 2x)}_{P(x)} y = 3 - 6x \quad \rightarrow \text{linear Eq.}$$

$$\mu(x) = e^{\int (1 - 2x) dx} = e^{x^2 - x}$$

$$\int \frac{d}{dx} \left[e^{x^2 - x} \cdot y \right] dx = \int (3 - 6x) e^{x^2 - x} dx$$

$$e^{x^2-x} \cdot y = -3 \int (2x-1) e^{x^2-x} dx$$

let $u = x^2 - x$
 $du = 2x-1 dx$

$$= -3 \int e^u du$$

$$e^{x^2-x} \cdot y = -3 e^u + C$$

$$\underbrace{e^{x^2-x}}_{\text{e}^{x^2-x}} \cdot y = \underbrace{-3 e^{x^2-x}}_{C^{x^2-x}} + C$$

$$y = -3 + \frac{C}{e^{x^2-x}}$$

$$\Rightarrow \boxed{y = -3 + C e^{x-x^2}}$$

Reduction of Order (idea: Reduce second order to
First order)

Case 1: if "y" (dependent variable) is missing from
the given equation.

$$\text{Form: } y'' = f(t, y')$$

$$\boxed{\begin{array}{l} \text{let } v = y' \\ v' = y'' \end{array}} \Leftrightarrow \frac{dy}{dt}$$

$$\text{Solve } y'' + t(y')^2 = 0 \quad \leftarrow \textcircled{z} \text{ } y \text{ is missing}$$

$$v' + tv^2 = 0$$

$$v' = -tv^2$$

$$\int \frac{1}{v^2} dv = \int -t dt$$

$$-v^{-1} = -\frac{t^2}{2} + C$$

$$-\frac{1}{v} = -\frac{t^2}{2} + C$$

$$\frac{1}{v} = \frac{t^2}{2} + C$$

$$\Rightarrow v = \frac{1}{\frac{t^2}{2} + C}$$

$$v = \frac{1}{\frac{t^2 + 2C}{2}}$$

$$v = \frac{2}{t^2 + c}$$

$$v = y' \Rightarrow y' = \frac{2}{t^2 + c}$$

$$\frac{dy}{dt} = \frac{2}{t^2 + c}$$

$$\int dy = \int \frac{2}{t^2 + c} dt$$

$$y = 2 \int \frac{1}{t^2 + (\sqrt{c})^2} dt$$

$$y = 2 \cdot \frac{1}{\sqrt{c}} \tan^{-1}\left(\frac{t}{\sqrt{c}}\right) + D$$

$$\begin{aligned} & \int \frac{1}{a^2 + x^2} dx \\ &= \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C \end{aligned}$$

$$\sqrt{c} = A$$

$$y(x) = \frac{2}{A} \tan^{-1}\left(\frac{t}{A}\right) + D$$

$$\underline{\text{eg2. Solve }} t^2 y'' + 2t y' - 1 = 0, \quad t > 0$$

Missing "y": Let $v = y'$ \Rightarrow $v' = y''$

$$t^2 v' + 2tv - 1 = 0$$

$$t^2 v' + 2tv = 1$$

$$\left\{ \begin{array}{l} v' + \frac{2}{t}v = \frac{1}{t^2} \\ \mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = t^2 \end{array} \right.$$

$$\Rightarrow \int \frac{d}{dt} (t^2 \cdot v) dt = \int 1 dt$$

$$t^2 v = t + C$$

$$v = \frac{1}{t} + C t^2$$

Replace $v = y'$

$$\frac{dy}{dt} = \frac{1}{t} + C t^2$$

$$\int 1 dy = \int \left(\frac{1}{t} + C t^2 \right) dt$$

$$y(t) = \ln t - C t^3 + C_2$$

$$\Rightarrow y(t) = \ln t + \frac{C_1}{t} + C_2$$

Case II: If "x" or "t" (independent) variable is missing. i.e. $y'' = f(y, y')$.

$$\text{let } \boxed{v = y'} = \frac{dy}{dt}$$

$$v' = y''$$

$$\frac{dv}{dt} = \frac{dv}{dy} \cdot \boxed{\frac{dy}{dt}}$$

$$\Rightarrow \boxed{\frac{dv}{dy} \cdot v = y''}$$

e.g. Solve $y \cdot y'' + (y')^2 = 0$.

$$\left. \begin{aligned} \text{let } v &= y' \\ \frac{dv}{dy} \cdot v &= y'' \end{aligned} \right\}$$

$$y \cdot \frac{dv}{dy} \cdot v + v^2 = 0$$

$$yv \frac{dv}{dy} = -v^2$$

$$y \frac{dv}{dy} = -v$$

$$y \cdot dv = -v dy$$

$$\int \frac{1}{v} dv = -\int \frac{1}{y} dy$$

$$\ln|v| = -\ln|y| + C$$

$$|v| = e^{-\ln|y|} \cdot e^C$$

$$|v| = C|y|^{-1}$$

$v > 0$,

$$v = \frac{c}{|y|}$$

$$v = y' = \frac{dy}{dt}$$

Replace

$$\frac{dy}{dt} = \frac{c}{|y|}$$

$$\int_{y>0} |y| dy = \int c dt$$

$$\int y dy = \int c dt$$

$$\frac{1}{2}y^2 = ct + D$$

$$y^2 = Ct + D$$

$$y = \pm \sqrt{Ct + D}$$