

(Review)

Math 39100 : Mar. 1 . 2023 : LECTURE 10

Exam 1: Monday, March 6 (1.2, 1.3, CH2, 3.1, 3.2, 3.3, 3.4)

WARM UP: Solve the following D.E:

$$1. \quad y'' - 4y' + 5y = 0$$

$$2. \quad y'' - 8y' + 16y = 0$$

$$3. \quad 2y'' - 7y' + 3y = 0$$

$$1. \quad r^2 - 4r + 5 = 0$$

$$a=1, \quad b=-4, \quad c=5$$

$$r = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{-4}}{2}$$

$$r = 2 \pm i$$

\downarrow \downarrow
 α β

$$\Rightarrow y = e^{\alpha t} (c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

$$y = e^{2t} (c_1 \cos(t) + c_2 \sin(t))$$

$$2. \quad y'' - 8y' + 16y = 0$$

$$r^2 - 8r + 16 = 0$$

$$(r-4)(r-4) = (r-4)^2 = 0$$

$r_1 = r_2 = 4$

$$y = C_1 e^{4t} + C_2 t e^{4t}$$

$$3. \quad 2y'' - 7y' + 3y = 0$$

$$2r^2 - 7r + 3 = 0$$

$$(2r-1)(r-3) = 0$$

$r_1 = \frac{1}{2}, \quad r_2 = 3$

$$y = C_1 e^{\frac{1}{2}t} + C_2 e^{3t}$$

§ 3.4 Cont.

Reduction of Order: (Given one of the solutions and looking for 2nd indep. soln).

y_1 Given .

Assume that General Soln : $y = v(t) y_1$

Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2y'' + 3ty' - y = 0, \quad t > 0,$$

find a fundamental set of solutions.

Assume that $y = v y_1 = v t^{-1}$

$$y' = v' t^{-1} - v t^{-2}$$

$$y'' = v'' t^{-1} - v' t^{-2} - \underbrace{v' t^{-2}}_{+2v t^{-3}} + 2v t^{-3}$$

$$y'' = v'' t^{-1} - 2v' t^{-2} + 2v t^{-3}$$

plus y , y' and y'' into the give eq:

$$2t^2(v'' t^{-1} - 2v' t^{-2} + 2v t^{-3}) + 3t(v' t^{-1} - v t^{-2}) - v t^{-1} = 0$$

$$2v'' t - 4v' + 4v t^{-1} + 3v' - 3v t^{-1} - v t^{-1} = 0$$

$$2v'' t - v' = 0$$

$$2w't - w = 0 \quad \text{let } w = v' \quad \text{(repar. or } w' = v'')$$

$$2t \frac{dw}{dt} = w \quad \text{integrating factor}$$

$$\int \frac{1}{w} dw = \int \frac{1}{2t} dt$$

$$\ln(w) = \frac{1}{2} \ln(t) + C \quad t > 0$$

$$\omega = C_1 e^{\frac{1}{2} C_0 t}$$

$$\Rightarrow \omega = C_1 t^{\frac{1}{2}}$$

$$\frac{dv}{dt} = C_1 t^{\frac{1}{2}}$$

$$\int dv = \int C_1 t^{\frac{1}{2}} dt$$

$$v = C_1 t^{\frac{3}{2}} + C_2$$

$$y = v \tilde{t}'$$

$$y = (C_1 t^{\frac{3}{2}} + C_2) \tilde{t}'$$

$$y = C_1 \underbrace{t^{\frac{3}{2}}}_{y_2} + C_2 \underbrace{\tilde{t}'}_{y_1} \quad \leftarrow \text{gen. Soln}$$

$$\Rightarrow \boxed{y_2 = t^{\frac{3}{2}}}$$

e.g. Find the second Indep. Soln to the given diff. eqn:

$$t^2 y'' - 2t y' - 10y = 0, \quad t > 0, \quad y_1 = t^5.$$

$$\text{Assume that } y = vt^5$$

$$y' = v't^4 + 5vt^4$$

$$y'' = v''t^5 + 10v't^4 + 20vt^3$$

$$v''t^7 + 10v't^6 + 20vt^5 - 2v't^6 - 10vt^5 - 10vt^5 = 0$$

$$v''t^7 + 8v't^6 = 0$$

$$v''t + 8v' = 0$$

$$\begin{aligned} \text{let } \omega &= v' \\ \omega' &= v'' \end{aligned}$$

$$\Rightarrow \omega' t + 8\omega = 0$$

$$\int \frac{1}{\omega} d\omega = - \int \frac{8}{t} dt$$

$$t > 0$$

$$\ln(\omega) = -8\ln(t) + C$$

$$\omega = C_1 e^{-8\ln t}$$

$$\Rightarrow \omega = C_1 t^{-8}$$

$$\int C_1 t^{-8} dt = \frac{C_1}{-7} t^{-7}$$

$$\int v' = \int C_1 t^{-8} dt$$

$$C_1 t^{-7}$$

$$V = C_1 t^{-7} + C_2$$

$$\text{Thus, } y = vt^5 = (c_1 t^7 + c_2) t^5$$

$$\Rightarrow y = \underbrace{c_1 t^7}_{y_2} + \underbrace{c_2 t^5}_{y_1}$$

So, $y_2 = t^7$

Exam 1: Monday, March 6 (1.2, 1.3, CH2, 3.1, 3.2, 3.3, 3.4)
Review past quizzes and HW problems

about 8-10 questions (multiple choice and short response)

- Classifications (order, linearity, ODE or PDE)
- Solving 1st order IVP



$$\mu(t) = e^{\int p(t) dt}$$

- Tank problems and Compound interest

$$\frac{dQ}{dt} = r_{in} C_{in} - r_{out} C_{out} \quad \mid \quad \frac{ds}{dt} = r s \pm k, \quad s(0) = s_0$$

$$C_{out} = \frac{Q(t)}{V_0 + (r_{in} - r_{out})t}$$

$Q(0)$ = initial condition

Reduction of Order (when y or t is missing)

Solving 2nd order IVP

Real distinct roots
Real repeated roots
Complex roots

Wronskian = $W(y_1, y_2) =$

$$\begin{vmatrix} + & - \\ y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 y'_2 - y_2 y'_1$$

Abel's theorem : $W = C e^{-\int p(t) dt}$

Determine the order, linear or nonlinear, PDE or ODE.

$$(1 + y^2) \frac{d^2y}{dt^2} + t \frac{dy}{dt} + y = e^t \quad \text{Order 2, nonlinear, ODE.}$$

Which of the following equations are exact?

I. $ye^{xy} + xe^{xy}y' = 0 \Rightarrow \underbrace{ye^{xy}}_M dx + \underbrace{xe^{xy}}_N dy = 0$

II. $\sin(y)dx = -x \cos(y)dy$

III. $2xye^{x^2}dx + xe^{x^2}dy = 0$

IV. $(ye^{-x} - xye^{-x})dx + (xe^{-x} + e^y)dy = 0$

$$u_{xx} + 2u_{tx} = 0$$

$$M_y = e^{xy} + xy e^{xy}$$

$$N_x = e^{xy} + xy e^{xy}$$

Solve the given initial value problem: $y'' - 8y' + 25y = 0$, $y(0) = 2$, $y'(0) = -2$. Then, find $y(\pi)$.

$$r = 4 \pm 3i$$

\downarrow
 $\sqrt{5}$

$$y = e^{4t} (C_1 \cos(3t) + C_2 \sin(3t)) \Rightarrow y(0) = 2 \Rightarrow 2 = 1 \cdot C_1 \cdot 1$$

$C_1 = 2$

$$y' = 4e^{4t} (C_1 \cos(3t) + C_2 \sin(3t)) + e^{4t} (-3C_1 \sin(3t) + 3C_2 \cos(3t))$$

$$y'(0) = -2 \Rightarrow -2 = 4C_1 + 3C_2$$

$$-2 = 4(2) + 3C_2$$

$$-10 = 3C_2 \Rightarrow C_2 = -\frac{10}{3}$$

$$\Rightarrow y = e^{4t} \left(2 \cos(3t) - \frac{10}{3} \sin(3t) \right)$$

$$y(\pi) = e^{4\pi} \left(2 \cos(3\pi) - \frac{10}{3} \sin(3\pi) \right)^0$$

$$y(\pi) = -2e^{4\pi}$$

A 500 gallon tank initially contains 100 gallons in which are dissolved 50 pounds of salt. The tank is flushed by pumping pure water into the tank at a rate of 3 gallons per minute and a well-mixed solution is pumped out at a rate of 1 gallon per minute. Which of the initial value problems below describes the quantity of salt, $Q(t)$, that would be in the tank at the time t , $0 < t < 200$.

With an initial investment of \$100, I invest \$500 per year, continuously, at an interest rate of 5%, leaving the money in the bank. Assuming the interest is compounded continuously, the initial value problem which describes the amount \$ S in the bank t years later is given by:

- (a) $S = (1.05)^t 500$, $S(0) = 100$.
- (b) $\frac{dS}{dt} = 500e^{0.05t}$, $S(0) = 100$.
- (c) $\frac{dS}{dt} = \frac{S}{20} + 500$, $S(0) = 100$.
- (d) $\frac{dS}{dt} = .05S + 100$, $S(0) = 500$.
- (e) $S = e^{0.05t} 500$, $S(0) = 100$.

Of the following S.O.L.D.E, which one(s) have complex roots of its characteristic polynomial.

- I. $y'' + 6y' + 9y = 0$
- II. $y'' - 6y' + 10y = 0$
- III. $y'' + y' - 6y = 0$
- IV. $y'' - 2y' + y = 0$

Which of the following represents the Wronskian of the pair of two linearly independent solutions of the given differential equation:

$$ty'' - 4y' + ty = 0, \quad t > 0$$

- I. $W(y_1, y_2) = Ce^{4t}$
- II. $W(y_1, y_2) = Ct^{-4}$
- III. $W(y_1, y_2) = Ct^4$
- IV. $W(y_1, y_2) = Ce^{-4t}$