

Math 39100 : Feb. 1. 2023 : LECTURE 3

Quiz 1: Monday, Feb. 6. 2023

(10 mins) on Sections 2.2, and 2.1 (beginning of class)
(Sep. eq., Homog. and linear)

§ 2.2 cont.

$$\int \frac{2-v}{(v+3)(v-1)} dv = \int \frac{-5/4}{v+3} dv + \int \frac{1/4}{v-1} dv$$
$$= -5/4 \ln|v+3| + \frac{1}{4} \ln|v-1|$$

$$-\frac{5}{4} \ln|v+3| + \frac{1}{4} \ln|v-1| = \ln|x| + C \quad \left(\begin{array}{l} \text{from} \\ \text{last} \\ \text{time} \end{array} \right)$$

$$\text{Sub in } v = \frac{y}{x}$$

$$\frac{1}{4} \ln \left| \frac{y}{x} - 1 \right| - \frac{5}{4} \ln \left| \frac{y}{x} + 3 \right| = \ln|x| + C$$

$$\frac{1}{4} \left(\ln \left| \frac{y-x}{x} \right| - 5 \ln \left| \frac{y+3x}{x} \right| \right) = \ln|x| + C$$

$$\boxed{\begin{aligned} \text{Recall: } C \ln x &= \ln x^C \\ \ln A - \ln B &= \ln(A/B) \end{aligned}}$$

$$\frac{1}{4} \left(\ln \left| \frac{y-x}{x} \right| - \ln \left| \frac{y+3x}{x} \right|^5 \right) = \ln(x) + C$$

$$\frac{1}{4} \left(\ln \left| \frac{(y-x)}{1} \cdot \frac{x^4}{(y+3x)^5} \right| \right) = \ln(x) + C$$

$$\frac{1}{4} \ln \left| \frac{x^4(y-x)}{(y+3x)^5} \right| - \ln(x) = C$$

$$\ln \left| \frac{x \cdot (y-x)^{1/4}}{(y+3x)^{5/4}} \cdot \frac{1}{x} \right| = C$$

$$\boxed{\ln \left| \frac{(y-x)^{1/4}}{(y+3x)^{5/4}} \right| = C}$$

Solve $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$. Separable? x

Homog. ✓

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2 \rightarrow \frac{x^2}{x^2} + \frac{xy}{x^2} + \frac{y^2}{x^2}$$

let $\boxed{v} = \frac{y}{x} \Rightarrow y = vx$

$$\boxed{\frac{dy}{dx}} = v + x \frac{dv}{dx}$$

$$v + x \frac{dv}{dx} = 1 + v + v^2$$

$$x \frac{dv}{dx} = 1 + v^2 \quad (\text{separable})$$

$$\int \frac{1}{1+v^2} dv = \int \frac{1}{x} dx$$

$$\tan^{-1}(v) = \ln|x| + C$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + C$$

OR

$$\tan^{-1}\left(\frac{y}{x}\right) - \ln|x| = C$$

How to check if an ODE is Homogeneous?

$$\text{Given } \frac{dy}{dx} = f(x, y)$$

assume it is not separable.

let $x \rightarrow xk$ } k is constant
 $y \rightarrow yk$

If $\frac{dy}{dx} = f(xk, yk) = f(x, y)$
 $\Rightarrow D.E$ is homogeneous.

e.g. Check if $x^2 dy - (x^2 + xy + y^2) dx = 0$ is homogeneous.

$$x^2 dy = (x^2 + xy + y^2) dx$$

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$$

$$\begin{aligned} x &\rightarrow kx \\ y &\rightarrow ky \end{aligned} \quad \frac{dy}{dx} = \frac{(kx)^2 + kx \cdot k \cdot y + (ky)^2}{(kx)^2} = \frac{k^2 x^2 + k^2 xy + k^2 y^2}{k^2 x^2}$$

$$= \frac{x^2 + xy + y^2}{x^2}$$

$$= f(x, y) \text{ original}$$

\therefore it's homogeneous.

§ 2.1 Linear D.E (Method of Integrating Factor)

Linear D.E in standard form :

$$\boxed{\frac{dy}{dx} + P(x)y = g(x)}$$

OR $\boxed{y' + p(x)y = g(x)}$

Solve the differential equation

$$(4 + t^2) \frac{dy}{dt} + 2ty = 4t.$$



We see $\frac{d}{dt} [(4+t^2)y] = 4t$

$$\int \frac{d}{dt} [(4+t^2)y] dt = \int 4t dt$$

$$(4+t^2)y = 2t^2 + C$$

"μ reads mō"

$$y = \frac{2t^2}{4+t^2} + \frac{C}{4+t^2}$$

Given $y' + p(x)y = g(x)$. [L.S.F]

We need a function, an integrating factor, denoted by

$\mu(x)$ so that when it is multiplied by LQ in

L.S.F we obtain the following:

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)g(x) \quad (4)$$

we want $\frac{d}{dx} [\mu(x)y]$ on the left hand side.

$$\mu'(x)y + \mu(x)y'$$

$$y\mu'(x) + y\mu(x)p(x) = \mu(x)y' + \mu(x)p(x)y$$

left
side
of
(4)

$$y\mu'(x) = \mu(x) \cdot p(x) y$$

$$\mu'(x) = \mu(x) p(x)$$

$$\frac{d\mu}{dx} = \mu(x) p(x) \quad (\text{separable})$$

$$\int \frac{1}{\mu(x)} d\mu = \int p(x) dx$$

$$\mu(x) > 0 \quad \ln \mu(x) = \int p(x) dx$$

$$\Rightarrow \boxed{\mu(x) = e^{\int p(x) dx}}$$

This is our integrating factor

□

Find the general solution of the differential equation

$$\frac{dy}{dt} + 2y = 4 - t \quad \text{already in L.S.F}$$

and plot the graphs of several solutions. Discuss the behavior of solutions as $t \rightarrow \infty$.

$$p(t) = -2$$

$$\mu(t) = e^{\int p(t) dt}$$

$$= e^{\int -2 dt}$$

$$\boxed{\mu(t) = e^{-2t}}$$

$$\frac{dy}{dt} \cdot e^{-2t} - 2e^{-2t}y = (4-t)e^{-2t}$$

[

$$\frac{d}{dt} \left[e^{-2t} y \right] = (4-t)e^{-2t}$$

$$\int \frac{d}{dt} \left(e^{-2t} y \right) dt = \int (4-t)e^{-2t} dt$$

$$e^{-2t} y = \int 4e^{-2t} dt - \int t e^{-2t} dt$$

$$\frac{e^{-2t}}{e^{-2t}} y = -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} + C$$

$$y = -2 + \frac{1}{2}t + \frac{1}{4} + Ce^{2t}$$

as $t \rightarrow \infty$, $y \rightarrow \infty$.

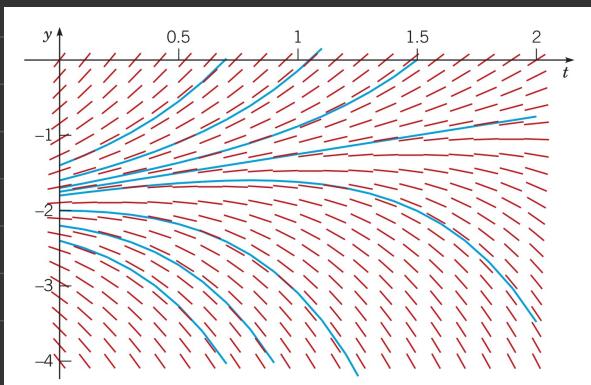
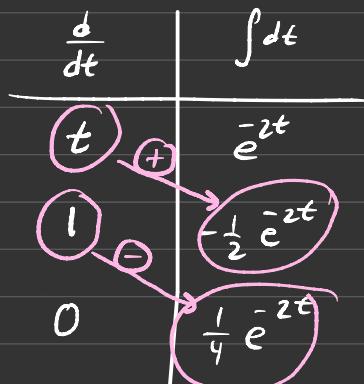


FIGURE 2.1.2 Direction field and integral curves of $y' - 2y = 4 - t$.

tabular method



Solve the initial value problem

$$ty' + 2y = 4t^2, \rightarrow y' + \frac{2}{t}y = 4t$$

$$y(1) = 2.$$

$$p(t) = \frac{2}{t}$$

Steps: i) If the D.E is linear, express it in S.F.

$$y' + p(x)y = g(x)$$

ii) identify $p(x)$ correctly.

iii) Find the integrating factor : $\mu(x) = e^{\int p(x) dx}$

(iv) multi the D.E in S.F by $\mu(x)$ (check that

the left hand side is $\frac{d}{dx} [\mu(x)y]$.

v) Integrate both sides.

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$t^2 \left[y' + \frac{2}{t}y = 4t \right]$$

$$\int \frac{d}{dt} [t^2 \cdot y] dt = \int 4t^3 dt$$

$$t^2 y = t^4 + c$$

$$y = t^2 + c t^{-2}$$

$$y(1) = 2.$$

$$2 = 1^2 + c \cdot 1^2$$

$$1 = c$$

$$y = t^2 + t^{-2}$$

Next time : Exact Method